

CLUSTER \mathcal{X} -VARIETIES FOR DUAL POISSON-LIE GROUPS I

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ABSTRACT. We associate a family of cluster \mathcal{X} -varieties to the dual Poisson-Lie group G^* of a complex semi-simple Lie group G of adjoint type given with the standard Poisson structure. This family is described by the W -permutohedron associated to the Lie algebra \mathfrak{g} of G : vertices being labeled by cluster \mathcal{X} -varieties and edges by new Poisson birational isomorphisms, on appropriate seed \mathcal{X} -tori, called *saltation*. The underlying combinatorics is based on a factorization of the Fomin-Zelevinsky twist maps into mutations and other new Poisson birational isomorphisms on seed \mathcal{X} -tori called *tropical mutations* (because they are obtained by a tropicalization of the mutation formula), associated to an enrichment of the combinatorics on double words of the Weyl group W of G .

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1. INTRODUCTION

The rising of the cluster combinatorics goes back to two sources: Berenstein, Fomin and Zelevinsky from one hand ([FZ02], [FZ03a], [BFZ05], [FZ07]) in their study of total positivity, and Fock and Goncharov on the other hand ([FG07a], [FG06b], [FG07b]) in their higher Teichmüller theory. These structures quickly spread to diverse mathematical areas such as: quiver representations, Poisson geometry, integrable systems, convex polytopes, tropical geometry, and so on. In this paper and its sequel [B], we use them to sharpen the geometry of dual Poisson-Lie groups. Therefore, according to the quantum duality principle [STS93], these two papers can be seen as the semi-classical starting point towards a cluster combinatorics describing the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ associated to a complex semi-simple Lie algebra \mathfrak{g} .

Let us recall that a Lie group G given with a Poisson structure is called *Poisson-Lie group* if the multiplication $m : G \times G \rightarrow G$ is a Poisson map, when the set $G \times G$ is given the Poisson product structure. Let $(\mathfrak{g}, \mathfrak{g}^*)$ be the tangent Lie bialgebra of G . According to the standard theory $(\mathfrak{g}^*, \mathfrak{g})$ is also a Lie bialgebra, and hence the Lie group associated with \mathfrak{g}^* is again a Poisson-Lie group called the *dual Poisson-Lie group* of G and denoted G^* . If the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ is factorizable, the Poisson-Lie group G^* can be embedded as a dense subset G_0 of G , when this one is given the appropriate Poisson structure. The symplectic leaves of G are then the G^* -orbits on G via the dressing transformations and the symplectic leaves of G^* are the conjugacy classes in G [STS85]. Let us denote this appropriate Poisson structure π_* when G is a complex semi-simple Lie group given with the standard Poisson structure π_G , that is a Sklyanin bracket associated to the standard r -matrix of the Belavin-Drinfeld classification. In that case, the dual of the Poisson-Lie group (G, π_G) may be identified with a subgroup in the direct product of two opposite Borel subgroups B and B_- of G , and we denote it (G^*, π_{G^*}) .

When G is a real split semisimple Lie group with trivial center, the geometry of (G, π_G) has been described by Fock and Goncharov via the combinatorics involved in cluster \mathcal{X} -variety. Recall that a cluster \mathcal{X} -variety is a Poisson variety obtained by gluing a set of tori along some specific bi-rational isomorphisms called (\mathcal{X}) -mutations. Each torus is given a log-canonical Poisson structure, that is a set of coordinates x_i and a skew-symmetric matrix $\widehat{\varepsilon}$, with generic integer values, such that $\{x_i, x_j\} = \widehat{\varepsilon}_{ij} x_i x_j$. Because mutations are Poisson maps relative to these log-canonical Poisson structures, cluster \mathcal{X} -varieties are naturally given a kind of Darboux coordinates. In [FG06b], using a coarser Poisson stratification of (G, π_G) into double Bruhat cells $G^{u,v}$, defined as the intersection of the cells BuB and B_-vB_- , where u, v belong to the Weyl group W of G , Fock and Goncharov have constructed canonical Poisson birational maps, called evaluation maps, of cluster \mathcal{X} -varieties into $(G^{u,v}, \pi_G)$ (one map $\text{ev}_{\mathbf{i}}$ for each seed \mathcal{X} -torus $\mathcal{X}_{\mathbf{i}}$ associated with a double reduced word \mathbf{i} associated to the pair $u, v \in W$); this construction provides for (G, π_G) a natural set of rational canonical coordinates. Canonical maps $\mu_{\mathbf{i} \rightarrow \mathbf{j}}$ associated with different double reduced words \mathbf{i}, \mathbf{j} are given by a composition of mutations simply related to the composition of generalized d -moves linking the double reduced words \mathbf{i} and \mathbf{j} .

In the present paper, using a key result of Evens and Lu [EL07], we adapt the construction of Fock and Goncharov to study the dual Poisson-Lie group (G^*, π_{G^*}) of (G, π_G) when G is a complex semi-simple of adjoint type. It turns that the description of (G^*, π_{G^*}) requires not one but a family of cluster \mathcal{X} -varieties indexed by the Weyl group W of G . This family is in fact described by the W -permutohedron associated to the Lie algebra \mathfrak{g} of G : vertices being labeled by cluster \mathcal{X} -varieties and edges by new Poisson birational isomorphisms, on appropriate seed \mathcal{X} -tori, called *saltation*. Roughly speaking, we associate to every cluster variety of this family a twisted evaluation, i.e. a composition of an evaluation map as above with a new map called twisted maps which generalizes the birational isomorphisms constructed by Evens and Lu in their study of Grothendieck resolutions [EL07], and use saltations to relate them. The combinatorics underlying this result rely on two new moves on double words added to the previous generalized d -moves; these moves are called τ -moves and *dual moves*. The maps on seed \mathcal{X} -tori associated to dual moves are saltations, whereas the maps on seed \mathcal{X} -tori associated to τ -moves are obtained by a tropicalization of the mutation formulas and therefore called *tropical mutations*. In fact, one of the key technical result here is an explicit factorization of the Fomin-Zelevinsky twist maps given in [FZ99] in terms mutations and tropical mutations.

In the sequel [B] of this paper, we will see how to use tropical mutations to include the De-Concini-Kac-Procesi Poisson automorphisms on (G_0, π_*) in the story.

Here is the organization of the paper. We fix the notation, give backgrounds on semi-simple Lie algebras and dual Poisson-Lie groups, and recall the basic definitions leading to the notion of cluster \mathcal{X} -variety in Section 2. We show, in a way useful for our purposes, how to naturally attach a cluster \mathcal{X} -variety $\mathcal{X}_{\mathbf{i}}$ to every double Bruhat cells $(G^{u,v}, \pi_G)$ via evaluation maps associated to any double (reduced) word \mathbf{i} in Section 3 (this section sums-up results of [FG06b]). In Section 4, we introduce new evaluation maps and new seeds related to double reduced words to state an analog of the previous construction of Fock and Goncharov for the dual Poisson-Lie group (G_0, π_*) ; although the result in this section are strongly generalized in Section 8, it is the occasion to give a flavor of our construction without using the machinery of generalized cluster transformations and saltations later developed. In Section 5, we enlarge the combinatorics on double words and on their related seed \mathcal{X} -tori by introducing respectively new moves called τ -moves and related

birational Poisson isomorphisms on seed \mathcal{X} -tori called tropical mutations; this enables us to describe the Fomin-Zelevinsky twist maps and their variations in terms on mutations and tropical mutations. In Section 6, we use the W -permutohedron associated to the Lie algebra \mathfrak{g} to study the combinatorics on double reduced words generated by generalized d -moves and enriched with tropical moves, as well as the related combinatorics on cluster \mathcal{X} -varieties; the idea is to prepare the ground for the cluster combinatorics related to twisted evaluations and dual Poisson-Lie groups, developed in Section 8. In Section 7, we generalized the results of Section 4: we introduce twisted evaluation and adapt the combinatorics of the previous section to get a family of cluster \mathcal{X} -varieties \mathcal{X}_w , associated to each element w of W parameterizing (BB_-, π_*) . (The results of Section 4 are rediscovered by setting $w = 1$.) In Section 8, we relate the previous twisted evaluations by cluster transformations and the birational Poisson isomorphisms called saltations; as a corollary, we get a parametrization of the dual Poisson Lie-group (BB_-, π_*) by a family of cluster \mathcal{X} -varieties; moreover, the cluster \mathcal{X} -varieties of this family are related by saltations described by the 1-skeleton of the W -permutohedron P_W . In Section 9, we start by giving an alternative way to describe twist maps with mutations and tropical mutations and provide evaluations for (G^*, π_{G^*}) in the spirit of the Kirillov-Reshetikhin multiplicative formula for the quantum R -matrix associated to $\mathcal{U}_q(\mathfrak{g})$; moreover, birational Poisson isomorphisms using to pass from the positive part to the negative part of (G^*, π_{G^*}) (and vice-versa) are easily encoded by paths on the 1-skeleton of W -permutohedron relating the identity and the longest element w_0 of W . Finally, we apply all our construction to the very special case of $G = SL(2, \mathbb{C})$ in Section 10, and, as a conclusion, we give the quantization of this elementary construction by considering the cluster combinatorics associated to the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$.

2. PRELIMINARIES

We fix the notation, give backgrounds on semi-simple Lie algebras and dual Poisson-Lie groups, and recall the basic definitions leading to the notion of cluster \mathcal{X} -variety.

2.1. Backgrounds on semi-simple Lie algebras. Let \mathfrak{g} be a complex semi-simple Lie algebra of rank l , A its Cartan matrix, and G its Lie group of adjoint type. Fix a Borel subgroup $B \subset G$, let B_- be the opposite Borel subgroup, $H = B \cap B_-$ the associated Cartan subgroup and N (resp. N_-) the unipotent radical of B (resp. B_-). Let $\mathfrak{h}, \mathfrak{n}, \mathfrak{n}_- \subset \mathfrak{g}$ be the Cartan and nilpotent subalgebras of \mathfrak{g} , corresponding respectively to H, N and N_- . In the following, we will denote $[1, l] := \{1, \dots, l\}$.

Let $\alpha_1, \dots, \alpha_l$ be the simple roots of \mathfrak{g} , and let $\omega_1, \omega_2, \dots, \omega_l \in \mathfrak{h}^*$ be the corresponding fundamental weights. For every $i \in [1, l]$, let (e_i, f_i, h_i) be the Chevalley generators of \mathfrak{g} ; they generate a Lie subalgebra \mathfrak{g}_{α_i} of \mathfrak{g} . In particular, we have $\omega_j(h_k) = \delta_{jk}$ for every $j, k \in [1, l]$. Let us recall that the *weight lattice* P is the set of all weights $\gamma \in \mathfrak{h}^*$ such that $\gamma(h_i) \in \mathbb{Z}$ for all i . So the group P has a \mathbb{Z} -basis formed by the fundamental weights. Every weight $\gamma \in P$ gives rise to a multiplicative character $a \mapsto a^\gamma$ of the maximal torus H ; this character is given by $\exp(h)^\gamma = e^{\gamma(h)}$, with $h \in \mathfrak{h}$.

The Lie algebra \mathfrak{g} being semi-simple, its Cartan matrix A is invertible and we can introduce a new basis $\{h^i, 1 \leq i \leq l\}$ on \mathfrak{h} putting

$$(2.1) \quad h^i := \sum (A^{-1})_{ij} h_j .$$

Let $D = \text{diag}(d_1, \dots, d_l)$ be the diagonal matrix associated with the set of Cartan symmetrizers; we put $\hat{a}_{ij} = d_i a_{ij} = a_{ij} d_j$. For every $x \in \mathbb{C}$ and $i \in [1, l]$, we define the group elements

$$E^i = \exp(e_i), \quad F^i = \exp(f_i), \quad H_i(x) = \exp(\log(x)h_i), \quad H^i(x) = \exp(\log(x)h^i)$$

related respectively to the generators e_i, f_i, h_i and h^i of \mathfrak{g} . The canonical inclusions $\varphi_i : \text{SL}(2, \mathbb{C}) \hookrightarrow G$ give in particular the following equalities for every nonzero complex number x .

$$(2.2) \quad \varphi_i \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = H^i(x) E^i H^i(x^{-1}), \quad \varphi_i \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = H^i(x^{-1}) F^i H^i(x),$$

$$(2.3) \quad \text{and} \quad \varphi_i \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix} = H_i(x) = \prod_{j \in [1, l]} H^j(x)^{a_{ij}}.$$

We denote by W the Weyl group of G . As an abstract group, W is a finite Coxeter group of rank l generated by the set of *simple reflections* $S = \{s_1, \dots, s_l\}$; it acts on $\mathfrak{h}^*, \mathfrak{h}$ and the Cartan subgroup H by

$$(2.4) \quad s_i(\gamma) = \gamma - \gamma(\alpha_i^\vee) \alpha_i, \quad s_i(h) = h - \alpha_i(h) \alpha_i^\vee \quad \text{and} \quad a^{w(\gamma)} = (\hat{w}^{-1} a \hat{w})^\gamma$$

for every $\gamma \in \mathfrak{h}^*, h \in \mathfrak{h}, w \in W$ and $a \in H$. Recall now that a *reduced word* for $w \in W$ is an expression for w in the generators belonging to S , which is minimal in length among all such expressions for w . Let us denote $\ell(w)$ this minimal length and $R(w)$ the set of reduced words associated to w . As usual, the notation w_0 will refer to the longest word of W .

Let us denote Π the set of positive roots of the Lie algebra \mathfrak{g} . It is well-known that if $i_1 \dots i_{\ell(w_0)}$ is a reduced expression for w_0 , then

$$\Pi = \{\alpha_{i_1}, s_{i_1}(\alpha_{i_2}), \dots, s_{i_1} \dots s_{i_{\ell(w_0)-1}}(\alpha_{i_{\ell(w_0)}})\},$$

each positive root occurring exactly one in the right-hand side. There are automorphisms T_1, \dots, T_l of \mathfrak{g} such that

$$(2.5) \quad \begin{aligned} T_i(e_i) &= -f_i, \quad T_i(f_i) = -e_i, \quad T_i(h_j) = h_j - a_{ji} h_i, \\ T_i(e_j) &= (-a_{ij})!^{-1} (\text{ad}_{e_i})^{-a_{ij}}(e_j), \quad \text{if } i \neq j, \\ T_i(f_j) &= (-a_{ij})!^{-1} (\text{ad}_{f_i})^{-a_{ij}}(f_j), \quad \text{if } i \neq j, \end{aligned}$$

where $\text{ad}_a(b) = [a, b]$ for every $a, b \in \mathfrak{g}$. To any positive root $\beta = s_{i_1} s_{i_2} \dots s_{i_{k-1}}(\alpha_{i_k}) \in \Pi$, $i_1 \dots i_{\ell(w_0)}$ being a reduced expression of the longest word w_0 of W , we associate the *positive* and *negative root vectors*

$$(2.6) \quad e_\beta = T_{i_1} T_{i_2} \dots T_{i_{k-1}}(e_{i_k}) \quad \text{and} \quad f_\beta = T_{i_1} \dots T_{i_{k-1}}(f_{i_k}).$$

Let us also recall that W can also be seen as the subgroup $\text{Norm}_G(H)/H$ of G . Thus, to every simple reflection $s_i \in S$ we associate the group element

$$\hat{s}_i = \varphi_i \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We can choose representatives in G for every element of W by setting $\widehat{w_1 w_2} = \widehat{w_1} \widehat{w_2}$ for every $w_1, w_2 \in W$ as long as $\ell(w_1) + \ell(w_2) = \ell(w_1 w_2)$. We finally have the *Bruhat decompositions* associated to G :

$$(2.7) \quad G = \bigcup_{u \in W} B \widehat{u} B = \bigcup_{v \in W} B_- \widehat{v} B_- = \bigcup_{w \in W} B \widehat{w} B_- .$$

2.2. Backgrounds on dual Poisson-Lie groups. A Lie group G given with a Poisson structure is called *Poisson-Lie group* if the multiplication $m : G \times G \rightarrow G$ is a Poisson map, when the set $G \times G$ is given the Poisson product structure. In the following, we will focus on the following Poisson-Lie groups.

2.2.1. The Poisson-Lie group (G, π_G) . According to the Belavin-Drinfel'd classification, the so-called *standard* classical r -matrix is given by the formula

$$r = \sum e_\alpha \wedge f_\alpha \in \mathfrak{g} \wedge \mathfrak{g} ,$$

where the summation is done over all positive roots. Let $\langle \cdot, \cdot \rangle$ be the Killing form on \mathfrak{g} . For every $x \in \mathfrak{g}$, $X \in G$ and every function $f \in \mathcal{F}(G)$, the left and right gradients are defined respectively by

$$\langle \nabla f(X), x \rangle = \frac{d}{dt} \Big|_{t=0} f(e^{tx} X) \quad \text{and} \quad \langle \nabla' f(X), x \rangle = \frac{d}{dt} \Big|_{t=0} f(X e^{tx}) .$$

If $f, g \in \mathcal{F}(G)$, let π_G be the following Poisson structure on G given by the *Sklyanin bracket* which transforms G into a Poisson-Lie group.

$$(2.8) \quad \{f, g\}_G = \frac{1}{2} (\langle \nabla f \otimes \nabla g, r \rangle - \langle \nabla' f \otimes \nabla' g, r \rangle) .$$

A Poisson stratification of (G, π_G) is obtained by using the Bruhat decompositions given by (2.7). As in [FZ99, Section 1.2], we associate to any $u, v \in W$ the *double Bruhat cell* $G^{u,v} \subset G$ defined by

$$G^{u,v} = B \widehat{u} B \cap B_- \widehat{v} B_- .$$

Each double Bruhat cell $G^{u,v}$ has dimension $\ell(u) + \ell(v) + l$ by [FZ99, Theorem 1.1]. The following result leads to the Poisson stratification

$$(2.9) \quad G = \bigcup_{u,v \in W} G^{u,v} .$$

Proposition 2.1. [HKKR00, KZ02, R03] *For every $u, v \in W$, the double Bruhat cells $G^{u,v}$ are the H -orbits, by the right-multiplication action, of the symplectic leaves of (G, π_G) .*

2.2.2. The Poisson-Lie group (G^*, π_{G^*}) . Let us recall that any multiplicative Poisson bracket on G identically vanishes at its unit element $e \in G$; its linearization at e gives rise to the structure of a Lie algebra on the dual space \mathfrak{g}^* ; multiplicativity then implies that the dual of the commutator map $[\cdot, \cdot] : \mathfrak{g}^* \times \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is a 1-cocycle on \mathfrak{g} . A pair $(\mathfrak{g}, \mathfrak{g}^*)$ with these properties is called a *Lie bialgebra*. Because of an equivalence between the category of Poisson-Lie groups (whose morphisms are Lie group homomorphisms which are also Poisson mappings) and the category of Lie bialgebras (whose morphisms are homomorphisms of Lie algebras such that their duals are homomorphisms of the dual algebras), finite dimensional Poisson-Lie groups always come by pair. The Poisson-Lie group associated to the Lie bialgebra $(\mathfrak{g}^*, \mathfrak{g})$ is called *the Poisson-Lie group dual to G* .

Let $\mathfrak{b}_\pm \subset \mathfrak{g}$ be the opposite Borel subalgebras associated to B_\pm . The dual Lie algebra \mathfrak{g}^* associated to the standard r -matrix may be identified with the following subalgebra of $\mathfrak{b}_+ \oplus \mathfrak{b}_-$:

$$\mathfrak{g}^* = \{(X_+, X_-) \in \mathfrak{b}_+ \oplus \mathfrak{b}_- \mid \text{diag } X_+ + \text{diag } X_- = 0\}.$$

Here, the application $\text{diag} : \mathfrak{g} \rightarrow \mathfrak{h}$ denotes the projection of \mathfrak{g} on its Cartan subalgebra. It can be lifted-up to give a projection from the Lie group G to its Cartan subgroup H , also denoted diag . The Lie group G^* associated with \mathfrak{g}^* may be identified with the following subgroup in $B_+ \times B_-$,

$$(2.10) \quad G^* = \{(b_+, b_-) \in B_+ \times B_- \mid \text{diag } b_+ \cdot \text{diag } b_- = I\}.$$

It carries a natural Poisson bracket which makes it a Poisson-Lie group; this is the dual Poisson-Lie group (G^*, π_{G^*}) of G . Let $t \in \mathfrak{g} \otimes \mathfrak{g}$ be the Casimir element of \mathfrak{g} , given by the following formula, and let us denote $r_\pm = r \pm t$.

$$t = \frac{1}{2} \left(\sum_{i,j \in [1,l]} (\hat{A}^{-1})_{ij} h_i \otimes h_j + \sum_{i \in [1,l]} (e_i \otimes f_i + f_i \otimes e_i) \right).$$

Proposition 2.2 ([STS85]). *Let us equip G with the Poisson structure π_* given by*

$$\{f, g\}_* = \frac{1}{2} (\langle \nabla f \otimes \nabla g, r \rangle + \langle \nabla' f \otimes \nabla' g, r \rangle) - \langle \nabla f \otimes \nabla' g, r_+ \rangle - \langle \nabla' f \otimes \nabla g, r_- \rangle.$$

The map $\phi : (G^, \pi_{G^*}) \rightarrow (BB_-, \pi_*) : (b_+, b_-) \mapsto b_+ b_-^{-1}$ is a Poisson covering of degree 2^l of Poisson manifolds.*

Proposition 2.3 ([STS85]). *The following conjugation action is a Poisson action.*

$$\begin{aligned} (G, \pi_G) \times (G, \pi_0) &\longrightarrow (G, \pi_0) \\ (g, h) &\longmapsto ghg^{-1} \end{aligned}$$

Following [EL07], let us now give a Poisson stratification for (G, π_*) . A *regular class function* on G is a regular function on G that is invariant under conjugation. Two elements $g_1, g_2 \in G$ are said to be in the same *Steinberg fiber* if $f(g_1) = f(g_2)$ for every regular class function f on G . For $t \in H$, let F_t be the Steinberg fiber containing t . By the Jordan decomposition of elements in G , every Steinberg fiber is of the form F_t for some $t \in H$. Moreover, the equality $F_{t'} = F_t$ is satisfied if and only if there exists $w \in W$ such that $t' = w(t)$, where W acts on H by the formula (2.4). The group G has therefore the decompositions

$$(2.11) \quad G = \bigcup_{t \in H, w \in W} F_{t,w} = \bigsqcup_{t \in H \setminus W, w \in W} F_{t,w} \quad \text{where} \quad F_{t,w} := B\hat{w}B_- \cap F_t.$$

Proposition 2.4. [EL07, Proposition 3.3] *For every $t \in H$ and $v \in W$:*

- $F_{t,w}$ is a non empty irreducible subvariety of G with dimension equal to $\dim(G) - l - \ell(w)$.
- $F_{t,v}$ is a finite union of H -orbits, for the conjugation action, of the symplectic leaves of (G, π_*) .

2.3. Backgrounds on cluster \mathcal{X} -varieties. We recall the definitions introduced by Fock and Goncharov underlying cluster \mathcal{X} -varieties. We add the notion of *erasing map*, which has been already used in [FG06b] without be named.

Definition 2.5. [FG07a, Definition 1.4] A *seed* \mathbf{I} is a quadruple (I, I_0, ε, d) where

- I is a finite set;
- $I_0 \subset I$;
- ε is a matrix ε_{ij} , $i, j \in I$, such that $\varepsilon_{ij} \in \mathbb{Z}$, unless $i, j \in I_0$;
- $d = \{d_i\}$, $i \in I$, is a subset of positive integers such that the matrix $\widehat{\varepsilon}_{ij} = \varepsilon_{ij}d_j$ is skew-symmetric.

Elements of the set I_0 are sometimes called *frozen vertices*. Here, we will not use this terminology, however, for the simple reason that, in the next, we will allow some birational Poisson isomorphisms in the direction of these frozen vertices, called tropical mutations. For every real number $x \in \mathbb{R}$, let us denote $[x]_+ = \max(x, 0)$ and

$$\text{sgn}(x) = \begin{cases} -1 & \text{if } x < 0 ; \\ 0 & \text{if } x = 0 ; \\ 1 & \text{if } x > 0 . \end{cases}$$

Definition 2.6. [FG07a, Section 1.2], [FZ02, Definition 4.2] Let $\mathbf{I} = (I, I_0, \varepsilon, d)$, $\mathbf{I}' = (I', I'_0, \varepsilon', d')$ be two seeds, and $k \in I \setminus I_0$. A *mutation in the direction k* is a map $\mu_k : I \longrightarrow I'$ satisfying the following conditions:

- $\mu_k(I_0) = I'_0$;
- $d'_{\mu_k(i)} = d_i$;
- $\varepsilon'_{\mu_k(i)\mu_k(j)} = \begin{cases} -\varepsilon_{ij} & \text{if } i = k \text{ or } j = k ; \\ \varepsilon_{ij} + \text{sgn}(\varepsilon_{ik})[\varepsilon_{ik}\varepsilon_{kj}]_+ & \text{if } i, j \neq k . \end{cases}$

Definition 2.7. [FG07a, Section 1.2] A *symmetry* on a seed $\mathbf{I} = (I, I_0, \varepsilon, d)$ is an automorphism σ of the set I preserving the subset I_0 , the matrix ε , and the numbers d_i . That is to say:

- $\sigma(I_0) = I_0$;
- $d_{\sigma(i)} = d_i$;
- $\varepsilon_{\sigma(i)\sigma(j)} = \varepsilon_{ij}$.

Let $|I|$ be the cardinal of every finite set I and $\mathbb{C}_{\neq 0}$ be the set of non-zero complex numbers.

Definition 2.8. [FG07a, Section 1.2] Let \mathbf{I} be a seed. The related *seed \mathcal{X} -torus* $\mathcal{X}_{\mathbf{I}}$ is the torus $(\mathbb{C}_{\neq 0})^{|I|}$ with the Poisson bracket

$$\{x_i, x_j\} = \widehat{\varepsilon}_{ij}x_i x_j,$$

where $\{x_i | i \in I\}$ are the standard coordinates on the factors. The *exchange part* of the seed \mathcal{X} -torus $\mathcal{X}_{\mathbf{I}}$ is the subtorus obtained by keeping only the x_j for $j \in I \setminus I_0$.

Symmetries and mutations on seeds induce involutive maps between the corresponding seed \mathcal{X} -tori, which are denoted by the same symbols μ_k and σ , and given by

- $x_{\sigma(i)} = x_i$
- $x_{\mu_k(i)} = \begin{cases} x_k^{-1} & \text{if } i = k ; \\ x_i x_k^{[\varepsilon_{ik}]_+} (1 + x_k)^{-\varepsilon_{ik}} & \text{if } i \neq k . \end{cases}$

Definition 2.9. [FG07a, Section 1.2] A *cluster transformation* linking two seeds (and two seed \mathcal{X} -tori) is a composition of symmetries and mutations. Let \mathbf{I} be a seed. A *cluster \mathcal{X} -variety* $\mathcal{X}_{|\mathbf{I}|}$ is obtained by taking every seed \mathcal{X} -torus obtained from \mathbf{I} by cluster transformations, and gluing them via the previous bi-rational isomorphisms.

Definition 2.10. [FG06b] Let $\mathbf{I} = (I, I_0, \eta, c)$ and $\mathbf{J} = (J, J_0, \varepsilon, d)$ be two seeds and let L be a set embedded into both I_0 and J_0 in a such a way that for any $i, j \in L$ we have $c(i) = d(i)$. Then the amalgamation of \mathbf{J} and \mathbf{I} is a seed $\mathbf{K} = (K, K_0, \zeta, b)$, such that $K = I \cup_L J$, $K_0 = I_0 \cup_L J_0$ and

$$\zeta_{ij} = \begin{cases} 0 & \text{if } i \in I \setminus L \text{ and } j \in J \setminus L ; \\ 0 & \text{if } i \in J \setminus L \text{ and } j \in I \setminus L ; \\ \eta_{ij} & \text{if } i \in I \setminus L \text{ and } j \in I \setminus L ; \\ \varepsilon_{ij} & \text{if } i \in J \setminus L \text{ and } j \in J \setminus L ; \\ \eta_{ij} + \varepsilon_{ij} & \text{if } i, j \in L . \end{cases}$$

This operation induces a homomorphism $\mathcal{X}_{\mathbf{J}} \times \mathcal{X}_{\mathbf{I}} \rightarrow \mathcal{X}_{\mathbf{K}}$ between the corresponding seed \mathcal{X} -tori given by the rule

$$z_i = \begin{cases} x_i & \text{if } i \in I \setminus L ; \\ y_i & \text{if } i \in J \setminus L ; \\ x_i y_i & \text{if } i \in L . \end{cases}$$

It is easy to check that it respects the Poisson structure and commutes with cluster transformations, thus is defined for the cluster \mathcal{X} -varieties, and not only for the seeds.

Definition 2.11. Let $\mathbf{I} = (I, I_0, \varepsilon, d)$ be a seed and $k \in I$. A *k -erasing map* on a seed $\mathbf{I} = (I, I_0, \varepsilon, d)$ is a morphism ς_k on \mathbf{I} such that

- $\varsigma_k(I_0) = I_0$ and $\varsigma_k(I) = I \setminus \{k\}$;
- $d_{\varsigma_k(i)} = d_i$;
- $\varepsilon_{\varsigma_k(i)\varsigma_k(j)} = \varepsilon_{ij}$.

erasing maps on seeds induce maps between the corresponding seed \mathcal{X} -tori, which are denoted by the same symbols ς_k , and given by

$$x_{\varsigma_k(i)} = x_i .$$

3. CLUSTER \mathcal{X} -VARIETIES RELATED TO (G, π_G)

We show, in a way useful for our purposes, how to naturally attach a cluster \mathcal{X} -variety $\mathcal{X}_{|\mathbf{i}|}$ to every double Bruhat cells $(G^{u,v}, \pi_G)$ via evaluation maps associated to any double (reduced) word \mathbf{i} . This section sums-up results of [FG06b].

3.1. Combinatorics on double words of W . We start by recalling the combinatorics on double words of W , which is derived from a well-known result of Tits. Following [FZ99], a (reduced) word of $W \times W$ is called a *double (reduced) word*. In order to avoid confusions we denote $\bar{1}, \dots, \bar{l}$ the indices of the reflections associated to the first copy of W , and $1, \dots, l$ the indices of the reflections associated to the second copy. A double (reduced) word of (u, v) is nothing else that a shuffle of a (reduced) word of u , written in the alphabet $[\bar{1}, \bar{l}]$, and of a (reduced) word of v , written in the alphabet $[1, l]$. We denote $D(u, v)$ and $R(u, v)$ the set of double words and double reduced words of (u, v) , respectively. (Therefore we have the inclusion $R(u, v) \subset D(u, v)$.) In particular, let $\mathbf{1} \in R(1, 1)$ be the double word associated to the unity of $W \times W$.

Let $w \in W$ and \mathbf{i} be a word of w . Following [BZ01, Section 7], we call a d -move (also named "braid-move" in [BB05]) a transformation of \mathbf{i} that replaces d consecutive entries i, j, i, j, \dots by j, i, j, i, \dots , for some i and j such that d is the order of $s_i s_j$, that is: if $a_{ij}a_{ji} = 0$ (resp. 1, 2, 3), then $d = 2$ (resp. 3, 4, 6). We call *nil-move* a transformation of \mathbf{i} that replaces the string ii by the elementary string i for some $i \in [1, l]$. The following result, called *the Tits theorem*, is standard and can be found, for example, in [BB05, Theorem 3.3.1].

Theorem 3.1. *Let (W, S) be a Coxeter group and $w \in W$.*

- Any expression for w can be transformed into a reduced expression for w by a sequence of *nil-moves* and *d-moves*.
- Every two reduced words for w can be connected by a sequence of *d-moves*.

Let us say that a letter i of \mathbf{i} is *positive* if $i \in [1, l]$ and *negative* if $i \in [\bar{1}, \bar{l}]$; a double word \mathbf{i} will be said to be *positive* (resp. *negative*) if all its letters are positive (resp. negative). Considering the group $W \times W$, we conclude that every two double reduced words $\mathbf{i}, \mathbf{j} \in R(u, v)$ can be obtained from each other by a sequence of *generalized d-moves*, listed in Figure 3.1. They contains

- *positive d-moves* for the alphabet $[1, l]$;
- *negative d-moves* for the alphabet $[\bar{1}, \bar{l}]$;
- *mixed 2-moves* that interchange two consecutive indices of opposite signs.

In the same way, any double word for the couple $(u, v) \in W \times W$ can be transformed to give any double reduced word of $R(u, v)$ by a sequence of *generalized dn-moves*, including

- *positive nil-moves* for the alphabet $[1, l]$;
- *negative nil-moves* for the alphabet $[\bar{1}, \bar{l}]$;
- *generalized d-moves*.

Positive and negative nil-moves are given by Figure 3.1.

$$\begin{array}{llll}
\text{or} & \dots i \bar{j} \dots & \rightsquigarrow & \dots \bar{j} i \dots \\
& \dots \bar{i} j \dots & \rightsquigarrow & \dots j \bar{i} \dots & \text{for every } i, j \in [1, l] \\
\\
\text{or} & \dots i j \dots & \rightsquigarrow & \dots j i \dots \\
& \dots \bar{i} \bar{j} \dots & \rightsquigarrow & \dots \bar{j} \bar{i} \dots & \text{when } a_{ij}a_{ji} = 0 \\
\\
\text{or} & \dots i j i \dots & \rightsquigarrow & \dots j i j \dots \\
& \dots \bar{i} \bar{j} \bar{i} \dots & \rightsquigarrow & \dots \bar{j} \bar{i} \bar{j} \dots & \text{when } a_{ij}a_{ji} = 1 \\
\\
\text{or} & \dots i j i j \dots & \rightsquigarrow & \dots j i j i \dots \\
& \dots \bar{i} \bar{j} \bar{i} \bar{j} \dots & \rightsquigarrow & \dots \bar{j} \bar{i} \bar{j} \bar{i} \dots & \text{when } a_{ij}a_{ji} = 2 \\
\\
\text{or} & \dots i j i j i j \dots & \rightsquigarrow & \dots j i j i j i \dots \\
& \dots \bar{i} \bar{j} \bar{i} \bar{j} \bar{i} \bar{j} \dots & \rightsquigarrow & \dots \bar{j} \bar{i} \bar{j} \bar{i} \bar{j} \bar{i} \dots & \text{when } a_{ij}a_{ji} = 3
\end{array}$$

FIGURE 1. The generalized d -moves

$$\begin{array}{ccc} \dots i \ i \ \dots & \rightsquigarrow & \dots i \ \dots \\ \text{and } \dots \bar{i} \ \bar{i} \ \dots & \rightsquigarrow & \dots \bar{i} \ \dots \end{array} \quad \text{for every } i \in [1, l]$$

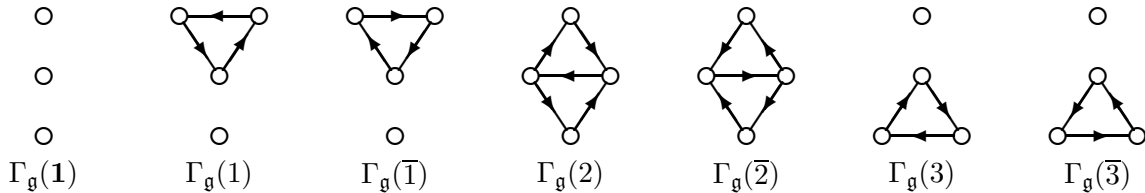
FIGURE 2. The positive and negative *nil*-moves

3.2. Quivers and seeds associated to a double word. We then reformulate the procedure to attach a seed to a double word given by [FG06b] via gluing on quivers in the spirit of [FST, Section 13].

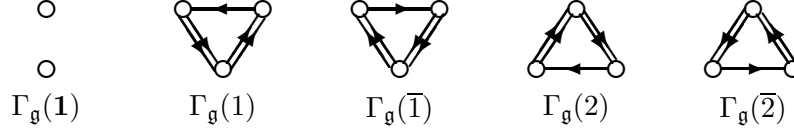
3.2.1. Dynkin quivers. Let $\Gamma_{\mathfrak{g}}$ be the Dynkin diagram of \mathfrak{g} , denote its vertex by $\binom{1}{0} \dots \binom{l}{0}$ and choose some $i \in [1, l]$. The *elementary Dynkin quiver* $\Gamma_{\mathfrak{g}}(i)$ is the directed graph obtained from Γ by the following procedure.

- Create a new vertex $\binom{i}{1}$ and call *i*-vertices the vertices $\binom{i}{0}, \binom{i}{1}$. and *j*-vertex the vertex $\binom{j}{0}$ for any $j \neq i$. The vertex $\binom{i}{0}$ and the *j*-vertices are then called *left outlets*, whereas $\binom{i}{1}$ and these *j*-vertices are called *right outlets*.
- Erase all the edges of the Dynkin diagram except the ones involving the vertex $\binom{i}{0}$.
- Rely the vertices $\binom{i}{1}$ to the vertex $\binom{i}{0}$ by an arrow such that $\binom{i}{1}$ is the tail of the arrow and $\binom{i}{0}$ its head.
- Create as many arrows between $\binom{i}{1}$ and each remaining vertices there are between $\binom{i}{0}$ and this remaining vertex. The heads and the tails of the arrows are directed in such a way that the triangle(s) thus created is/are oriented.

The elementary Dynkin quiver $\Gamma_{\mathfrak{g}}(\bar{i})$ is then obtained from $\Gamma_{\mathfrak{g}}(i)$ by reversing the orientation of all the arrows. To these elementary Dynkin quivers, we add the *trivial Dynkin quiver* $\Gamma_{\mathfrak{g}}(\mathbf{1})$ obtained from the Dynkin diagram by removing all the edges; as above the vertices of this quiver are labeled by $\binom{i}{0}$, with $i \in [1, l]$. (For the trivial Dynkin quiver $\Gamma_{\mathfrak{g}}(\mathbf{1})$, the set of left outlets is, by definition, the set of right outlets.) Figure 3 and Figure 4 describe respectively the elementary Dynkin quivers of the cases $\mathfrak{g} = A_3$ and $\mathfrak{g} = B_2$. Let us stress that in all our examples, outlets will be marked by unfilled circles. Other conventions in our drawing will be the following: the type of vertices is given by a kind of height function where vertices of type 1 are at the top of the quiver and vertices of type l at the bottom; moreover, left outlets will always be drawn at the left of right outlets if both are *k*-vertices but different.

FIGURE 3. Elementary Dynkin quivers for $\mathfrak{g} = A_3$

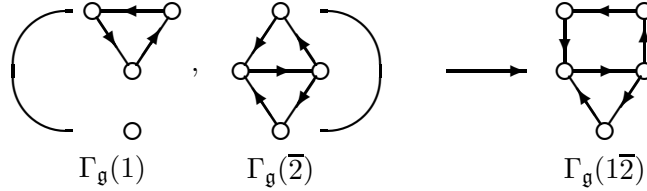
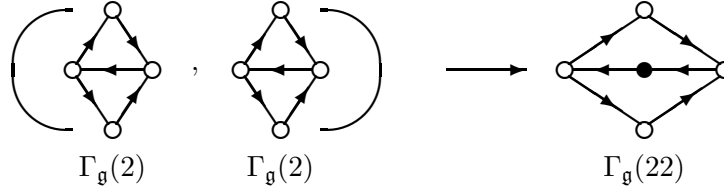
A quiver Γ is called a *Dynkin quiver* if it can be obtained from a collection of disjoint elementary Dynkin quivers, coming from the same Dynkin diagram, by the following procedure, called *amalgamation*. Let $\mathbf{i} = i_1 \dots i_n$ be a double word and $\Gamma_{\mathfrak{g}}(i_1), \dots, \Gamma_{\mathfrak{g}}(i_n)$ be the associated elementary Dynkin quivers. For every $k \in [1, l]$, we put a total order, called the *k*-order on the set of the *k*-vertices of all the elementary Dynkin quivers in such

FIGURE 4. Elementary Dynkin quivers for $\mathfrak{g} = B_2$

a way that $\binom{i_j}{0} < \binom{i_j}{1}$ for every j , and that every k -vertices of $\Gamma_{\mathfrak{g}}(i_l)$ are lower than the k -vertices of $\Gamma_{\mathfrak{g}}(i_j)$ when $l < j$.

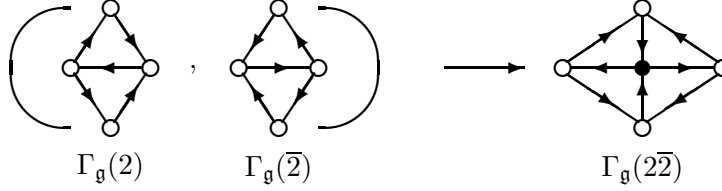
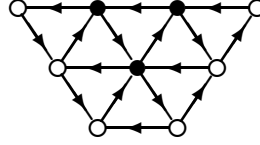
- For every $j \in [1, n-1]$, glue every right outlets of $\Gamma_{\mathfrak{g}}(i_j)$ to every left outlets of $\Gamma_{\mathfrak{g}}(i_{j+1})$ in such a way that k -vertices are glued together;
- relabel each k -vertex following the k -order, that is $\binom{k}{i}$ is the i^{th} k -vertex, with an increasing numbering, of the set of all k -vertices given by the k -order;
- redefine the set of outlets: left outlets (resp. right outlets) are vertices $\binom{k}{i}$ such that $\binom{k}{i} \leq \binom{k}{j}$ (resp. $\binom{k}{j} \leq \binom{k}{i}$) for every j ;
- if Γ contains a pair of edges connecting the same pair of vertices but going in opposite directions, then remove each such a pair of edges; (As a result, all arrows linking two vertices point now in the same direction.)
- for every $k_1 \neq k_2$, erase arrows linking k_1 -vertices to k_2 -vertices until there is not more arrows linking these vertices in the resulting graph than edges linking the k_1^{th} vertex and the k_2^{th} -vertex in the corresponding Dynkin diagram.

The amalgamation process is certainly easier to figure out on examples. Figure 5, Figure 6 and Figure 7 describe some amalgamations in the case $\mathfrak{g} = A_3$.

FIGURE 5. The amalgamation $(\Gamma_{\mathfrak{g}}(1), \Gamma_{\mathfrak{g}}(\bar{2})) \mapsto \Gamma_{\mathfrak{g}}(1\bar{2})$ for $\mathfrak{g} = A_3$ FIGURE 6. The amalgamation $(\Gamma_{\mathfrak{g}}(2), \Gamma_{\mathfrak{g}}(2)) \mapsto \Gamma_{\mathfrak{g}}(22)$ for $\mathfrak{g} = A_3$

The resulting graph is called the *Dynkin quiver* $\Gamma_{\mathfrak{g}}(\mathbf{i})$. Therefore, a Dynkin quiver is associated to every double word \mathbf{i} . In particular, every elementary Dynkin quiver is a Dynkin quiver, and the elementary Dynkin quiver $\Gamma_{\mathfrak{g}}(1)$ does not affect a gluing.

It is easy to see that the amalgamation is associative. In particular, the Dynkin quivers $\Gamma_{\mathfrak{g}}(\mathbf{i})$ and $\Gamma_{\mathfrak{g}}(\mathbf{j})$ can be amalgamated to obtained the Dynkin quiver $\Gamma_{\mathfrak{g}}(\mathbf{ij})$. Let us also remark that for every vertex i of a Dynkin quiver Γ , there exists $k \in [1, l]$ such that i is a k -vertex. Such a k is called the *vertex-type* of i and is denoted $k(i)$. It is clear that

FIGURE 7. The amalgamation $(\Gamma_{\mathfrak{g}}(2), \Gamma_{\mathfrak{g}}(\bar{2})) \mapsto \Gamma_{\mathfrak{g}}(2\bar{2})$ for $\mathfrak{g} = A_3$ FIGURE 8. The quiver $\Gamma_{\mathfrak{g}}(123121)$ for $\mathfrak{g} = A_3$

the vertex-type remains unchanged by the amalgamation. Moreover, for every $j \in [1, l]$, we denote $N^j(\mathbf{i})$ the number of vertices in $\Gamma_{\mathfrak{g}}(\mathbf{i})$ whose vertex-type is j . Stated otherwise, $N^j(\mathbf{i})$ is the number of times the letter j or \bar{j} appears in the double word \mathbf{i} .

3.2.2. Seeds associated to double words. We derive the seeds constructed by Fock and Goncharov from the previous Dynkin quivers in the following way. Let \mathbf{i} be a double word, and Γ be the associated Dynkin quiver. Let $\bar{B}(\mathbf{i}) = (\bar{b}_{ij})$ denote the skew-symmetric matrix whose rows and columns are labeled by the vertices of Γ , and whose entry \bar{b}_{ij} is equal to the number of edges going from i to j minus the number of edges going from j to i . The skew-symmetrizable matrix $B(\Gamma) = (b_{ij})$ is obtained from $\bar{B}(\mathbf{i})$ by the following skew-symmetrizing formula, involving the vertex-types $k(i)$ and $k(j)$, of i and j :

$$d_{k(i)} b_{ij} = \bar{b}_{ij} = -\bar{b}_{ji} = -b_{ji} d_{k(j)} ,$$

where d_1, \dots, d_l are the set of non-zero natural numbers that symmetrize the Cartan matrix. Let \mathbf{i} be a double word and $\Gamma_{\mathfrak{g}}(\mathbf{i})$ be its associated Dynkin quiver. The seed $\mathbf{I}(\mathbf{i}) = (I(\mathbf{i}), I_0(\mathbf{i}), \varepsilon(\mathbf{i}), d(\mathbf{i}))$ is defined in the following way.

- The set $I(\mathbf{i})$ is the set of vertices of $\Gamma_{\mathfrak{g}}(\mathbf{i})$, with a partial order induced by the k -orders on the set of k -vertices, the sets $I_0^{\mathfrak{R}}(\mathbf{i}), I_0^{\mathfrak{L}}(\mathbf{i})$ are respectively the set of right outlets and left outlets of $\Gamma_{\mathfrak{g}}(\mathbf{i})$, and $I_0(\mathbf{i})$ is the set of outlets. Stated otherwise, the set $I(\mathbf{i})$ (resp. $I_0^{\mathfrak{L}}(\mathbf{i}), I_0^{\mathfrak{R}}(\mathbf{i})$ and $I_0(\mathbf{i})$) is the set of all ordered pairs $\binom{j}{k}$ such that $j \in [1, l]$, and $0 \leq k \leq N^j(\mathbf{i})$ (resp. $k = 0$, $k = N^j(\mathbf{i})$, and $k \in \{0, N^j(\mathbf{i})\}$), where $N^j(\mathbf{i})$ is the number of times the letter j or \bar{j} appears in \mathbf{i} .
- The matrix $\varepsilon(\mathbf{i})$ is given by a normalisation of $B(\mathbf{i})$:

$$(3.1) \quad \varepsilon(\mathbf{i})_{kl} = \begin{cases} \frac{B(\mathbf{i})_{kl}}{2} & \text{for } k \in I_0(\mathbf{i}) \text{ and } l \in I_0(\mathbf{i}); \\ B(\mathbf{i})_{kl} & \text{otherwise.} \end{cases}$$

- The multiplier $d(\mathbf{i})$ is given by associating to every vertex j the Cartan symetrizer of its type-vertex, that is:

$$d(\mathbf{i})_j = d_{k(j)} .$$

It is easy to translate the amalgamation procedure at the level of seeds: for every double words \mathbf{i}, \mathbf{j} , the *amalgamated seed* $(I(\mathbf{ij}), I_0(\mathbf{ij}), \varepsilon(\mathbf{ij}), d(\mathbf{ij}))$ is defined in the following way [FG06b]. The elements of the set $d(\mathbf{ij})$ are equal to the corresponding Cartan symmetrizer as above and the matrix $\varepsilon(\mathbf{ij})$ is given by

$$(3.2) \quad \varepsilon(\mathbf{ij})_{\binom{i}{k}\binom{j}{l}} = \begin{cases} \varepsilon(\mathbf{i})_{\binom{i}{k}\binom{j}{l}} & \text{if } k < N^i(\mathbf{i}) \text{ and } l < N^j(\mathbf{j}) ; \\ \varepsilon(\mathbf{i})_{\binom{i}{k}\binom{j}{l}} + \varepsilon(\mathbf{j})_{\binom{i}{0}\binom{j}{0}} & \text{if } k = N^i(\mathbf{i}) \text{ and } l = N^j(\mathbf{j}) ; \\ \varepsilon(\mathbf{j})_{\binom{i}{k-N^i(\mathbf{i})}\binom{j}{l-N^j(\mathbf{j})}} & \text{if } k > N^i(\mathbf{i}) \text{ and } l > N^j(\mathbf{j}) ; \\ 0 & \text{otherwise .} \end{cases}$$

In particular, for every $i \in [1, l]$ and $\mathbf{i} \in \{\mathbf{i}, \bar{\mathbf{i}}\}$, the matrices $\varepsilon(\bar{\mathbf{i}})$ and $\varepsilon(\mathbf{i})$ have their entries labeled by the elements of $I(i) = I(\bar{i})$ and are given by the following equalities and zero otherwise.

$$(3.3) \quad \varepsilon(\mathbf{i})_{\binom{i}{1}\binom{j}{0}} = \frac{a_{ij}}{2} = -\varepsilon(\bar{\mathbf{i}})_{\binom{i}{1}\binom{j}{0}}, \quad \varepsilon(\bar{\mathbf{i}})_{\binom{i}{1}\binom{j}{0}} = -\frac{a_{ij}}{2} = -\varepsilon(\mathbf{i})_{\binom{i}{1}\binom{j}{0}} .$$

3.3. Cluster \mathcal{X} -varieties related to (G, π_G) . We attach a seed \mathcal{X} -torus to each of the previous seeds. We use them to describe the combinatorics underlying the Poisson geometry of double Bruhat cells, as in [FG06b]. Let us recall the basis of \mathfrak{h} given by (2.1) and the related group elements $H^i(x)$ given by (2.3). For every $\mathbf{i} \in \{\mathbf{1}, i, \bar{i}\}$ we denote $\mathcal{X}_{\mathbf{i}}$ the seed \mathcal{X} -torus associated to the elementary seed $(I(\mathbf{i}), I_0(\mathbf{i}), \varepsilon(\mathbf{i}), d(\mathbf{i}))$ and $\text{ev}_{\mathbf{i}} : \mathcal{X}_{\mathbf{i}} \rightarrow G$ the related evaluation map given by

$$\begin{aligned} \text{ev}_{\mathbf{1}} : \mathbb{C}_{\neq 0}^l &\longrightarrow G & : \quad \left(x_{\binom{1}{0}}, \dots, x_{\binom{i}{0}}, \dots, x_{\binom{l}{0}}\right) &\longmapsto \prod_j H^j(x_{\binom{j}{0}}) , \\ \text{ev}_i : \mathbb{C}_{\neq 0}^{l+1} &\longrightarrow G & \left(x_{\binom{1}{0}}; \dots, x_{\binom{i}{0}}, x_{\binom{i}{1}}, x_{\binom{i+1}{0}}, \dots, x_{\binom{l}{0}}\right) &\longmapsto \prod_j H^j(x_{\binom{j}{0}}) E^i H^i(x_{\binom{i}{1}}) , \\ \text{ev}_{\bar{i}} : \mathbb{C}_{\neq 0}^{l+1} &\longrightarrow G & \left(x_{\binom{1}{0}}; \dots, x_{\binom{i}{0}}, x_{\binom{i}{1}}, x_{\binom{i+1}{0}}, \dots, x_{\binom{l}{0}}\right) &\longmapsto \prod_j H^j(x_{\binom{j}{0}}) F^i H^i(x_{\binom{i}{1}}) . \end{aligned}$$

Proposition 3.2. [FG06b, Proposition 3.11] *For every $i \in [1, l]$, $\mathbf{i} \in \{\mathbf{1}, i, \bar{i}\}$ and $(u, v) \in W \times W$ such that $\mathbf{i} \in R(u, v)$, the evaluation map $\text{ev}_{\mathbf{i}} : \mathcal{X}_{\mathbf{i}} \rightarrow (G^{u,v}, \pi_G)$ is a Poisson birational isomorphism on a Zariski open set of $G^{u,v}$.*

Lemma 3.3. [FG06b] *The amalgamation procedure induces a Poisson homomorphism $\mathbf{m} : \mathcal{X}_{\mathbf{i}} \times \mathcal{X}_{\mathbf{j}} \rightarrow \mathcal{X}_{\mathbf{ij}}$ between the corresponding seed \mathcal{X} -tori given by*

$$(3.4) \quad z_{\binom{i}{k}} = \begin{cases} x_{\binom{i}{k}} & \text{if } 0 \leq k < N^i(\mathbf{i}) ; \\ x_{\binom{i}{k}} y_{\binom{i}{0}} & \text{if } k = N^i(\mathbf{i}) ; \\ y_{\binom{i}{k-N^i(\mathbf{i})}} & \text{if } N^i(\mathbf{i}) < k \leq N^i(\mathbf{i}) + N^i(\mathbf{j}) , \end{cases}$$

where x_i, y_j and z_k are the associated variables.

Now, let $\mathbf{i} = i_1 \dots i_k$ be a double word, $\mathcal{X}_{\mathbf{i}}$ be the seed \mathcal{X} -torus given by the associated amalgamation $\mathbf{m} : \mathcal{X}_{i_1} \times \dots \times \mathcal{X}_{i_k} \rightarrow \mathcal{X}_{\mathbf{i}}$, and \mathbf{z} be the amalgamated variable $\mathbf{m}(\mathbf{x}_1, \dots, \mathbf{x}_k)$. We define the *evaluation map*

$$(3.5) \quad \text{ev}_{\mathbf{i}} : \mathcal{X}_{\mathbf{i}} \rightarrow G : \mathbf{z} \mapsto \text{ev}_{i_1}(\mathbf{x}_1) \dots \text{ev}_{i_k}(\mathbf{x}_k) \quad \text{where} \quad \mathbf{z} = \mathbf{m}(\mathbf{x}_1, \dots, \mathbf{x}_k) .$$

Using the multiplicative property of the Poisson-Lie group (G, π_G) , we see that this evaluation is also a Poisson map. So Proposition 3.2 leads to the Poisson statement of the following result.

Theorem 3.4 ([FG06b]). *For any $u, v \in W$ and $\mathbf{i} \in R(u, v)$ the map $\text{ev}_{\mathbf{i}} : \mathcal{X}_{\mathbf{i}} \rightarrow (G^{u,v}, \pi_G)$ is a Poisson birational isomorphism onto a Zariski open set of the double Bruhat cell $G^{u,v}$.*

We now introduce cluster transformations in the framework. Let us say that a double reduced word of length d is d -minimal if we can perform a generalized d -move on it. Let us call n -minimal the double words i and i for every $i \in [1, l] \cup [\bar{1}, \bar{l}]$. Finally, a double word is said to be dn -minimal if it is d -minimal or n -minimal. To any two dn -minimal double words \mathbf{i} and \mathbf{i}' related by a generalized dn -move $\delta : \mathbf{i} \mapsto \mathbf{i}'$, we associate a cluster transformation $\mu_{\mathbf{i} \rightarrow \mathbf{i}'} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{i}'}$ in the following way:

$$(3.6) \quad \mu_{\mathbf{i} \rightarrow \mathbf{i}'} = \begin{cases} \varsigma_{(1)}^{(i)} \circ \mu_{(1)}^{(i)}, & \text{if } \delta \text{ is a nil-move;} \\ \mu_{(1)}^{(i)}, & \text{if } \delta \text{ is a move } i \bar{i} \leftrightarrow \bar{i} i \text{ or a 3-move;} \\ \mu_{(1)}^{(i)} \mu_{(j)}^{(i)} \mu_{(1)}^{(i)}, & \text{if } \delta \text{ is a 4-move;} \\ \mu_{(2)}^{(j)} \mu_{(1)}^{(i)} \mu_{(1)}^{(j)} \mu_{(2)}^{(i)} \mu_{(2)}^{(j)} \mu_{(1)}^{(i)} \mu_{(2)}^{(j)} \mu_{(1)}^{(i)} \mu_{(2)}^{(j)}, & \text{if } \delta \text{ is a 6-move;} \\ \text{the identity map} & \text{otherwise,} \end{cases}$$

where, as in [FG06b], we have denoted an expression $\mu_{\mu_i(j)} \mu_i$ by $\mu_j \mu_k$, an expression $\mu_{\mu_{\mu_i(j)} \mu_i(k)} \mu_{\mu_i(j)} \mu_i$ by $\mu_k \mu_j \mu_i$, and so on.

Since mutations commute with amalgamation, we may extend these definitions to any two double words $\mathbf{i}, \mathbf{i}' \in D(u, v)$ related by a generalized dn -move. Finally, if \mathbf{i}, \mathbf{j} are double words linked by a sequence $\delta_{\mathbf{i} \rightarrow \mathbf{j}}$ of generalized dn -moves and $\mathbf{i} \rightarrow \mathbf{i}_1 \rightarrow \cdots \rightarrow \mathbf{i}_{n-1} \rightarrow \mathbf{j}$ is the associated chain of elements, we define the cluster transformation $\mu_{\mathbf{i} \rightarrow \mathbf{j}}$ as the composition $\mu_{\mathbf{i}_{n-1} \rightarrow \mathbf{j}} \circ \cdots \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}_1}$. The following result is easily derived from Theorem 3.1.

Lemma 3.5. *A double reduced word $\mathbf{j} \in R(u, v)$ can be obtained from a double reduced word $\mathbf{i} \in R(u', v')$ by a sequence of generalized d -moves $\delta_{\mathbf{i} \rightarrow \mathbf{j}}$ if and only if the equalities $u' = u$ and $v' = v$ are satisfied.*

Let $u, v \in W$ and $\mathbf{i}, \mathbf{j} \in R(u, v)$. Because the birational Poisson isomorphism $\mu_{\mathbf{i} \rightarrow \mathbf{j}} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{j}}$, associated to a sequence $\delta_{\mathbf{i} \rightarrow \mathbf{j}}$ of generalized d -moves, is a cluster transformation, we will denote $\mathcal{X}^{u,v}$ the cluster \mathcal{X} -variety associated to the set $R(u, v)$. Stated otherwise, to any double reduced word $\mathbf{i} \in R(u, v)$ corresponds a local chart $(\mathcal{X}_{\mathbf{i}}, \varphi_{\mathbf{i}})$ in the cluster \mathcal{X} -variety $\mathcal{X}^{u,v}$, and the cluster transformation $\mu_{\mathbf{i} \rightarrow \mathbf{j}}$ is the transition map between the local charts $(\mathcal{X}_{\mathbf{i}}, \varphi_{\mathbf{i}})$ and $(\mathcal{X}_{\mathbf{j}}, \varphi_{\mathbf{j}})$ for any $\mathbf{j} \in R(u, v)$. Let us also denote $\iota_{\mathbf{i}} : R(u, v) \rightarrow \mathbf{i}$ every time we choose the element $\mathbf{i} \in R(u, v)$. The diagrams in Figure 9 are therefore commutative.

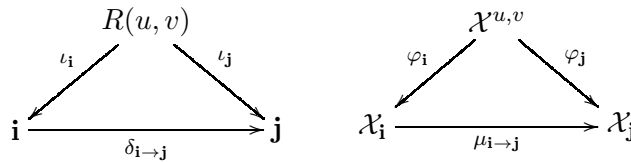


FIGURE 9. The set $R(u, v)$ and the cluster \mathcal{X} -variety $\mathcal{X}^{u,v}$

We finally give the way to attach the cluster \mathcal{X} -variety $\mathcal{X}^{u,v}$ to the double Bruhat cell $(G^{u,v}, \pi_G)$, for every $u, v \in W$.

Lemma 3.6 ([FG06b]). *For any $u, v \in W$ and $\mathbf{i}, \mathbf{j} \in D(u, v)$ such that \mathbf{j} is obtained from \mathbf{i} by a sequence of generalized dn -moves, we have $\text{ev}_{\mathbf{i}} = \text{ev}_{\mathbf{j}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}}$.*

Theorem 3.7 ([FG06b]). *For any $u, v \in W$ and $\mathbf{i}, \mathbf{j} \in R(u, v)$, we have $\text{ev}_{\mathbf{i}} = \text{ev}_{\mathbf{j}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}}$.*

The cluster \mathcal{X} -variety $\mathcal{X}^{u,v}$ is therefore attached to the double Bruhat cell $(G^{u,v}, \pi_G)$ for every $u, v \in W$. Let us denote \mathcal{X} the application which associate to any double word \mathbf{i} the corresponding seed \mathcal{X} -torus $\mathcal{X}_{\mathbf{i}}$. We then sum-up Theorem 3.4 and Theorem 3.7 by abusively saying that there exists a Poisson map $\text{ev}^{u,v} : \mathcal{X}^{u,v} \rightarrow (G^{u,v}, \pi_G)$.

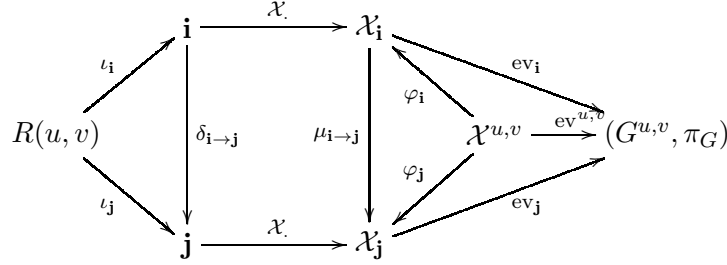


FIGURE 10. The cluster \mathcal{X} -variety $\mathcal{X}^{u,v}$ associated to $(G^{u,v}, \pi_G)$

4. TRUNCATION MAPS AND CLUSTER \mathcal{X} -VARIETIES RELATED TO (G, π_*)

We introduce new evaluation maps and new seeds related to double reduced words to state analogs of Theorem 3.4 and Theorem 3.7 for the dual Poisson-Lie group $G^* \subset (G, \pi_*)$. Although the result in this section will be strongly generalized in Section 8, it is the occasion to introduce truncation maps and to give a flavor of what will be later called twisted evaluations and its related combinatorics, without using the machinery of generalized cluster transformations and saltations that we start to develop in Section 5.

4.1. Double reduced Bruhat cells and reduced evaluations. According to [BZ01, Section 4.3], let $L^{u,v}$ be the *reduced double Bruhat cell* associated to every $u, v \in W$, that is: the quotient of double Bruhat cell $G^{u,v}$ by the H -right multiplication:

$$(4.1) \quad L^{u,v} = G^{u,v} / H.$$

We are going to slightly modify the cluster \mathcal{X} -varieties previously constructed in order to evaluate these double reduced Bruhat cells. For every double word \mathbf{i} , let $\varsigma_{\mathfrak{R}, \mathbf{i}}$ be the erasing map associated to the set $I_0^{\mathfrak{R}}(\mathbf{i})$ of right outlets, defined as the product over the set $I_0^{\mathfrak{R}}(\mathbf{i})$ of the erasing maps ς_j given by Definition 2.11.

$$(4.2) \quad \varsigma_{\mathfrak{R}, \mathbf{i}} = \prod_{j \in I_0^{\mathfrak{R}}(\mathbf{i})} \varsigma_j.$$

We denote \mathbf{i}^{red} the image of the seed $\mathbf{I}(\mathbf{i})$ by $\varsigma_{\mathfrak{R}, \mathbf{i}}$, and $\mathcal{X}_{\mathbf{i}}^{\text{red}}$ the seed \mathcal{X} -torus associated to the seed \mathbf{i}^{red} . Therefore, we have $\varsigma_{\mathfrak{R}, \mathbf{i}} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{i}}^{\text{red}}$. Every cluster transformation $\mu_{\mathbf{i} \rightarrow \mathbf{j}} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{j}}$ canonically leads to a cluster transformation between the associated seed \mathcal{X} -tori $\mu_{\mathbf{i} \rightarrow \mathbf{j}}^{\text{red}} : \mathcal{X}_{\mathbf{i}}^{\text{red}} \rightarrow \mathcal{X}_{\mathbf{j}}^{\text{red}}$ by the relation

$$\mu_{\mathbf{i} \rightarrow \mathbf{j}}^{\text{red}} \circ \varsigma_{\mathfrak{R}, \mathbf{i}} = \varsigma_{\mathfrak{R}, \mathbf{j}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}}.$$

Definition 4.1. Let \mathbf{i} be a double reduced word. It is clear that the map defined on $\mathcal{X}_{\mathbf{i}}$ and given by

$$\mathbf{x} \mapsto \text{ev}_{\mathbf{i}}(\mathbf{x}) \prod_{j \in [1, l]} H^j(x_{\binom{-1}{N^j(\mathbf{i})}})$$

doesn't depend on x_k when $k \in I_0^{\mathfrak{R}}(\mathbf{i})$, thus we get an evaluation $\text{ev}_{\mathbf{i}}^{\text{red}} : \mathcal{X}_{\mathbf{i}}^{\text{red}} \rightarrow G$ called a *reduced evaluation*. Stated in a rougher way, the reduced evaluation associated to the double word \mathbf{i} is obtained from the evaluation map $\text{ev}_{\mathbf{i}}$ by setting the cluster variables x_j to 1 (or forgetting the related Cartan element $H^i(x_j)$) for every $j \in I_0^{\mathfrak{R}}(\mathbf{i})$.

When the double word \mathbf{i} is a double reduced word, Theorem 3.4 and Theorem 3.7 are easily adapted to reduced double Bruhat cells using reduced evaluations. We get the following result.

Corollary 4.2. *For any $u, v \in W$ and $\mathbf{i} \in R(u, v)$ the map $\text{ev}_{\mathbf{i}}^{\text{red}} : \mathcal{X}_{\mathbf{i}}^{\text{red}} \rightarrow (L^{u, v}, \pi_G)$ is a Poisson birational isomorphism onto a Zariski open set of the double Bruhat cell $L^{u, v}$. And for any $u, v \in W$ and $\mathbf{i}, \mathbf{j} \in R(u, v)$, we have $\text{ev}_{\mathbf{i}}^{\text{red}} = \text{ev}_{\mathbf{j}}^{\text{red}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}}^{\text{red}}$.*

Remark 4.3. This corollary remains valid even if the Lie group G is not of adjoint type but simply connected.

4.2. Truncation maps on cluster \mathcal{X} -varieties. The cluster \mathcal{X} -variety we are going to associate to the dual Poisson-Lie group (BB_-, π_*) in the next subsection can be easily obtained from the cluster \mathcal{X} -variety \mathcal{X}^{w_0, w_0} of Section 3 by the notions of truncation map and truncated cluster \mathcal{X} -varieties we are going to introduce now. The underlying idea is to force the apparition of the Casimir of (BB_-, π_*) . We start by giving the general setting.

Definition 4.4. Let $\mathbf{I} = (I, I_0, \varepsilon, d)$ be a seed and $J \subset I$. The *truncation map associated to J* is a map $\mathbf{t}_J : \mathbf{I} \rightarrow \mathbf{I}_J$ such that the seed $\mathbf{I}_J = (I, I_0, \varepsilon', d)$ is given by:

$$(4.3) \quad \varepsilon'_{ij} = \begin{cases} \varepsilon_{ij} & \text{if } i, j \in I \setminus J; \\ 0 & \text{otherwise.} \end{cases}$$

To any finite set J , let us denote \mathcal{X}_J^0 the seed \mathcal{X} -torus associated to a seed $(J, J, 0, d)$. Let $\mathbf{I} = (I, I_0, \varepsilon, d)$ be a seed such that $J \subset I$, and $\mathbf{I}' = (I, I_0, \varepsilon', d)$ be the image of \mathbf{I} by \mathbf{t}_J . To every $\mathbf{t} \in \mathcal{X}_J$, we associate the subtorus $\mathcal{X}_{\mathbf{I}'}(\mathbf{t}) \subset \mathcal{X}_{\mathbf{I}'}$ constituted of elements $\mathbf{x} \in \mathcal{X}_{\mathbf{I}'}$ such that $x_i = t_i$ for every $i \in J$. It is a Poisson subtorus because of the formula (4.3). The map \mathbf{t}_J induces an homomorphism $\mathbf{t}_{J(t)} : \mathcal{X}_{\mathbf{I}} \rightarrow \mathcal{X}_{\mathbf{I}'}(\mathbf{t})$ associated to any $\mathbf{t} \in \mathcal{X}_J^0$ and given by

$$x_{\mathbf{t}_{J(t)}(i)} = \begin{cases} x_i & \text{if } i \in I \setminus J; \\ t_i & \text{if } i \in J. \end{cases}$$

In particular, the *trivial truncated map*, associated to the empty set $J = \emptyset$, is the identity map. Now, let $\mathbf{I} = (I, I_0, \varepsilon, d)$ be a seed and J be a subset of I . To any cluster transformation $\phi_{\mathbf{I} \rightarrow \mathbf{J}} : \mathcal{X}_{\mathbf{I}} \rightarrow \mathcal{X}_{\mathbf{J}}$, we associate the following Poisson birational isomorphism $\phi_{\mathbf{I}_J \rightarrow \mathbf{J}_J} : \mathcal{X}_{\mathbf{I}_J} \rightarrow \mathcal{X}_{\mathbf{J}_J}$, called *truncated cluster transformation* and given by

$$x_{\phi_{\mathbf{I}_J \rightarrow \mathbf{J}_J}(i)} = \begin{cases} x_{\phi_{\mathbf{I} \rightarrow \mathbf{J}}(i)} & \text{if } i \in I \setminus J; \\ x_i & \text{if } i \in J. \end{cases}$$

It is clear that for every $\mathbf{t} \in \mathcal{X}_J^0$ this map admits a restriction $\phi_{\mathbf{I}_J \rightarrow \mathbf{J}_J} : \mathcal{X}_{\mathbf{I}_J}(\mathbf{t}) \rightarrow \mathcal{X}_{\mathbf{J}_J}(\mathbf{t})$ which is also a birational Poisson isomorphism.

We would like to define truncation maps at the level of cluster \mathcal{X} -varieties. A sufficient condition is given by the following immediate result.

Lemma 4.5. *Let $\mathbf{I} = (I, I_0, \varepsilon, d)$ be a seed and $J \subset I_0$. The following equality is satisfied for every cluster transformation $\phi_{\mathbf{I} \rightarrow \mathbf{J}} : \mathcal{X}_{\mathbf{I}} \rightarrow \mathcal{X}_{\mathbf{J}}$ and every $t \in \mathcal{X}_J^0$.*

$$(4.4) \quad \phi_{\mathbf{I}_J \rightarrow \mathbf{J}_J} \circ \mathbf{t}_{J(t)} = \mathbf{t}_{\phi_{\mathbf{I} \rightarrow \mathbf{J}}(J)(t)} \circ \phi_{\mathbf{I} \rightarrow \mathbf{J}} .$$

Remark 4.6. The formula (4.4) is not necessarily true if the condition $J \subset I_0$ is omitted. Indeed, consider the seed $\mathbf{I}(i)$ for any $i \in [1, l]$, and the set $J = \{\binom{i}{1}\}$, with the cluster transformation $\mu_{\binom{i}{1}}$. In fact, a generalization of the formula (4.4) will be the starting point for the definition of the Poisson birational isomorphisms called saltations and given in Subsection 8.2.

Definition 4.7. Let $\mathbf{I} = (I, I_0, \varepsilon, d)$ be a seed, J be a subset of I , and \mathbf{I}' be the truncated seed associated to the seed \mathbf{I} and the set J . The *truncated cluster \mathcal{X} -variety* $\mathcal{X}_{|\mathbf{I}'|}$ of the cluster \mathcal{X} -variety $\mathcal{X}_{|\mathbf{I}|}$ is obtained by taking every seed \mathcal{X} -torus obtained from $\mathcal{X}_{\mathbf{I}'}$ by cluster transformations, and gluing them as usual. Moreover, if we denote $\mathcal{X}_{|\mathbf{I}'|}(t)$ the cluster \mathcal{X} -variety obtained from every seed \mathcal{X} -torus $\mathcal{X}_{\mathbf{I}'}(t) \subset \mathcal{X}_{\mathbf{I}'}$, we have the following Poisson stratification, because of (4.3).

$$\mathcal{X}_{|\mathbf{I}'|} = \bigcup_{t \in \mathcal{X}_J^0} \mathcal{X}_{|\mathbf{I}'|}(t) .$$

The truncated map $\mathbf{t}_{J(t)}$ is therefore well-defined at the level of cluster \mathcal{X} -varieties for every fixed $t \in \mathcal{X}_J^0$ if J is a subset of I_0 . It is denoted $\mathbf{t}_{J(t)} : \mathcal{X}_{|\mathbf{I}|} \rightarrow \mathcal{X}_{|\mathbf{t}_{J(t)}(\mathbf{I})|}(t)$.

We now focus on particular truncations on cluster \mathcal{X} -varieties associated to double words. For every double word \mathbf{i} , let $[\mathbf{i}]_{\mathfrak{R}}$ be the image of the seed $\mathbf{I}(\mathbf{i})$ by the *right truncation map* $\mathbf{t}_{\mathbf{i}_{\mathfrak{R}}}$ associated to the set of right outlets $I_0^{\mathfrak{R}}(\mathbf{i})$ of $I(\mathbf{i})$. Stated otherwise, for every double word \mathbf{i} , the seed $[\mathbf{i}]_{\mathfrak{R}} = (I(\mathbf{i}), I_0(\mathbf{i}), \eta(\mathbf{i}), d(\mathbf{i}))$ is the seed defined by the values

$$(4.5) \quad \eta(\mathbf{i})_{ij} = \begin{cases} \varepsilon(\mathbf{i})_{ij} & \text{if } i, j \in I(\mathbf{i}) \setminus I_0^{\mathfrak{R}}(\mathbf{i}); \\ 0 & \text{otherwise.} \end{cases}$$

An example of right truncation map is given by Figure 11. The *right truncated seed \mathcal{X} -*

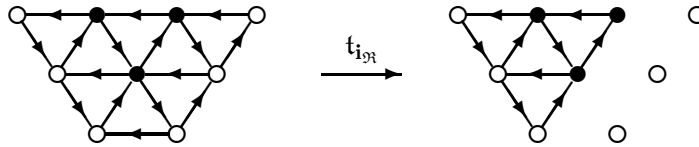


FIGURE 11. The right truncation map $\mathbf{t}_{\mathbf{i}_{\mathfrak{R}}} : \Gamma_{A_3}(\mathbf{i}) \rightarrow \Gamma_{A_3}([\mathbf{i}]_{\mathfrak{R}})$ for $\mathbf{i} = 123121$

torus $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}(t)$ associated to any $t \in H$ is the subset of $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$ obtained by fixing the following cluster variables $\mathbf{x}(\mathfrak{R})$ associated to right outlets

$$(4.6) \quad \mathbf{x}(\mathfrak{R}) = \{x_j | j \in I_0^{\mathfrak{R}}(\mathbf{i})\}$$

via the equality

$$(4.7) \quad \text{ev}_1(x_{\binom{1}{N^1(\mathbf{i})}}, \dots, x_{\binom{l}{N^l(\mathbf{i})}}) = t .$$

Equation (4.5) implies that $\mathcal{X}_{\mathbf{i}}(t)$ is a Poisson submanifold of $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$ and according to Definition 4.4 the right truncation map $\mathbf{t}_{\mathfrak{R}}$ on seeds induces *right truncation maps* on seed \mathcal{X} -tori $\mathbf{t}_{\mathfrak{R}}(t) : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}(t)$. The following result is clear.

Lemma 4.8. *Let \mathbf{i} be a double word. The right truncated seed \mathcal{X} -torus $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$ is Poisson isomorphic to the direct product $\mathcal{X}_{\mathbf{i}}^{\text{red}} \times \mathcal{X}_{\mathbf{1}}$ of seed \mathcal{X} -tori. Moreover, the following equality is satisfied for any double word \mathbf{j} obtained from \mathbf{i} by a sequence of generalized d -moves.*

$$(4.8) \quad \begin{aligned} \mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{j}]_{\mathfrak{R}}} : \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}} &\rightarrow \mathcal{X}_{[\mathbf{j}]_{\mathfrak{R}}} \\ x_{\mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{j}]_{\mathfrak{R}}}(i)} &= \begin{cases} x_{\mu_{\mathbf{i} \rightarrow \mathbf{j}}(i)} & \text{if } i \in I \setminus J ; \\ x_i & \text{if } i \in J . \end{cases} \end{aligned}$$

It implies in particular that the cluster transformation $\mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{j}]_{\mathfrak{R}}}$ sends $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}(t)$ to $\mathcal{X}_{[\mathbf{j}]_{\mathfrak{R}}}(t)$ for every \mathbf{j} linked to \mathbf{i} by composition of generalized d -moves. Finally, the cluster \mathcal{X} -variety $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$, constructed from the seed \mathcal{X} -torus $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$ associated to a double word \mathbf{i} is called a *right truncated cluster \mathcal{X} -variety*. By Lemma 4.5, there exists a *truncation map* $\mathbf{t}_{\mathfrak{R}}$ associating to every cluster \mathcal{X} -variety $\mathcal{X}_{[\mathbf{i}]}$ its right truncated cluster \mathcal{X} -variety $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$:

$$\mathbf{t}_{\mathfrak{R}} : \mathcal{X}_{[\mathbf{i}]} \mapsto \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}} .$$

4.3. Dual evaluations and first cluster \mathcal{X} -varieties related to (G, π_*) . We use the previous truncated cluster \mathcal{X} -varieties to start the geometrical combinatorics of the Poisson manifold (G, π_*) . For every element $x \in B_-B$, let $x = [x]_-[x]_0[x]_+$ be its Gauss decomposition, that is: $[x]_{\pm}$ belongs to the unipotent parts N_{\pm} of the respective Borel subgroups B_{\pm} and $[x]_0$ to the Cartan part H of G . We will also consider the following notations.

$$(4.9) \quad [x]_-[x]_{\geq 0} = [x]_-[x]_0[x]_+ = [x]_{\leq 0}[x]_+ .$$

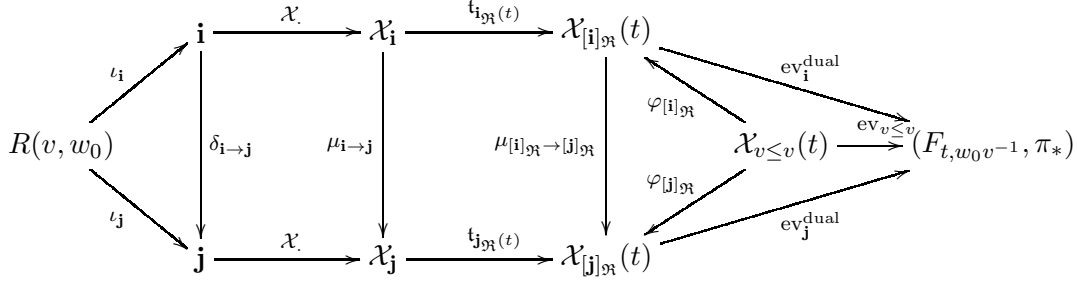
Definition 4.9. Let $v \in W$. For every $\mathbf{i} \in R(v, w_0)$, we define the *dual evaluation map* $\text{ev}_{\mathbf{i}}^{\text{dual}} : \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}} \rightarrow G$ by the formula:

$$(4.10) \quad \text{ev}_{\mathbf{i}}^{\text{dual}}(\mathbf{x}) = \text{ev}_{\mathbf{i}}(\mathbf{x}) \widehat{w_0} [\text{ev}_{\mathbf{i}}^{\text{red}}(\mathbf{x}) \widehat{w_0}]_{\leq 0}^{-1} .$$

Dual evaluations will be generalized into twisted evaluations in Subsection 7.2. (We refer to Remark 7.4 for more details.) For the moment, let us remember the Poisson stratification (2.7) of (G, π_*) .

Theorem 4.10. *For every $v \in W$, $t \in H$ and $\mathbf{i} \in R(v, w_0)$, the map $\text{ev}_{\mathbf{i}}^{\text{dual}} : \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}(t) \rightarrow (F_{t, w_0 v^{-1}}, \pi_*)$ is a Poisson birational isomorphism onto a Zariski open set $F_{t, w_0 v^{-1}}^0$ of $F_{t, w_0 v^{-1}}$.*

Theorem 4.10 will be deduced from Theorem 7.9. The synthesis diagram, given by Figure 10 and relating, for every $u, v \in W$, the cluster \mathcal{X} -variety $\mathcal{X}^{u, v}$ to the double Bruhat cell $G^{u, v}$ can therefore be adapted to get a cluster \mathcal{X} -variety $\mathcal{X}_{v \leq v}(t)$ associated to $(F_{t, w_0 v^{-1}}, \pi_*)$, for every $v \in W$ and every $t \in H$. It is illustrated in Figure 12. (The weird terminology for " $\mathcal{X}_{v \leq v}(t)$ " and " $\text{ev}_{v \leq v}$ " will be explained in Subsection 7.3.) Finally, for every double word \mathbf{i} , let $\mathcal{X}_{\mathbf{i}}^{\text{dual}} \subset \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$ be such that the elements of the set of variables $\mathbf{x}(\mathfrak{R})$ are pairwise disjoint. It is a Poisson submanifold of $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$ because of (4.5). Thus, the following corollaries are respectively deduced from the second decomposition of (2.11) with Theorem 4.10, and Theorem 3.7 with Lemma 4.8 and equation (4.10).

FIGURE 12. The cluster \mathcal{X} -variety $\mathcal{X}_{v \leq v}(t)$ associated to $(F_{t, w_0 v^{-1}}, \pi_*)$

Corollary 4.11. *For every $\mathbf{i} \in R(w_0, w_0)$, the map $\text{ev}_{\mathbf{i}}^{\text{dual}} : \mathcal{X}_{\mathbf{i}}^{\text{dual}} \rightarrow (BB_-, \pi_*)$ is a Poisson birational isomorphism on a Zariski open set of BB_- .*

Corollary 4.12. *For any $v \in W$ and $\mathbf{i}, \mathbf{j} \in R(v, w_0)$, we have $\text{ev}_{\mathbf{i}}^{\text{dual}} = \text{ev}_{\mathbf{j}}^{\text{dual}} \circ \mu_{[i]_{\mathfrak{R}} \rightarrow [j]_{\mathfrak{R}}}$.*

5. τ -MOVES, TROPICAL MUTATIONS, AND TWIST MAPS

We enlarge the combinatorics on double words and on their related seed \mathcal{X} -tori by introducing respectively new moves based on the involution $i \mapsto \bar{i}$ and new birational Poisson isomorphisms on seed \mathcal{X} -tori obtained by a tropicalization of the mutation formula. This enables us to describe the Fomin-Zelevinsky twist maps and their variations in terms on cluster transformations and tropical mutations.

5.1. τ -moves and tropical mutations. We introduce τ -moves and *tropical mutations*. From now on, we suppose that for every seed $\mathbf{I} = (I, I_0, \varepsilon, d)$, the related matrix ε is such that ε_{ij} is a rational number, for every $i, j \in I$.

5.1.1. The right and left τ -moves. We now consider the map $i \mapsto \bar{i}$ as an involution on $[1, l] \cup [\bar{l}, \bar{1}]$. Let us first enrich the combinatorics on double words.

Definition 5.1. For every double word $\mathbf{i} = i_1 \dots i_n$, let $\mathfrak{L}(\mathbf{i})$ (resp. $\mathfrak{R}(\mathbf{i})$) be the double word obtained by changing the first letter (resp. last letter) i of \mathbf{i} into \bar{i} :

$$\mathfrak{L}_{i_1}(\mathbf{i}) = \bar{i}_1 i_2 \dots i_n \quad \text{and} \quad \mathfrak{R}_{i_n}(\mathbf{i}) = i_1 \dots i_{n-1} \bar{i}_n.$$

The map $\mathbf{i} \mapsto \mathfrak{L}_{i_1}(\mathbf{i})$ (resp. $\mathbf{i} \mapsto \mathfrak{R}_{i_n}(\mathbf{i})$) is called a *left* (resp. *right*) τ -move on \mathbf{i} . (We will simply denote these maps $\mathbf{i} \mapsto \mathfrak{L}(\mathbf{i})$ and $\mathbf{i} \mapsto \mathfrak{R}(\mathbf{i})$ when no confusion occurs.)

Remark 5.2. The set $R(u, v)$ of double reduced words associated to the elements $u, v \in W$ is generally stable by neither left nor right τ -moves. Even the set of all the double reduced words associated to W is stable by neither left nor right τ -moves. However it contains subsets that are stable. For example, consider the following set $R^\tau(w)$ associated to any $w \in W$.

$$R^\tau(w) = \bigcup_{w' \leq w \in W} R(w'^{-1}, ww'^{-1}).$$

It is stable by right τ -moves and will be used to describe the combinatorics associated to the dual Poisson-Lie group (G^*, π_{G^*}) .

5.1.2. *Tropical mutations.* The τ -moves on double words lead to a new type of mutations on the associated seed \mathcal{X} -tori, called tropical mutations and defined in the following way. We first identify \mathbb{Q} with the Cartesian product $\mathbb{Z} \times \mathbb{N} \setminus \{0\}$, by decomposing every element into its numerator p and its positive denominator q with $(p, q) = 1$, and associate to every value ε_{ij} of ε its numerator b_{ij} . So we have $b_{ij} = \varepsilon_{ij}$ unless $i, j \in I_0$; let us, from now on, suppose that the denominator q is the same for every $i, j \in I_0$. In particular, we recall that for every double word \mathbf{i} we have

$$(5.1) \quad b(\mathbf{i})_{kl} = \begin{cases} 2\varepsilon(\mathbf{i})_{kl} & \text{for } k, l \in I_0 ; \\ \varepsilon(\mathbf{i})_{kl} & \text{otherwise .} \end{cases}$$

Then we need an additional data of the set of outlets of a seed.

Definition 5.3. Let $\mathbf{I} = (I_0, I, \varepsilon, d)$ be a seed such that I_0 is not empty. A *cover* \mathfrak{C} on \mathbf{I} is a family of sets $I_1, \dots, I_n \subset I_0$ such that $I_0 = \cup_{i=1}^n I_i$. (The union is not necessary disjoint.) For every $k \in I_0$, we denote $I_0(k)$ the union

$$I_0(k) := \bigcup_{\{i|k \in I_i\}} I_i .$$

Remark 5.4. Every seed $\mathbf{I} = (I_0, I, \varepsilon, d)$ with a non empty set I_0 comes from two trivial covers, where the first one is obtained by setting $I_1 := I_0$, and, on the opposite way, where the second one is given by the union $I_0 = \cup_{k \in I_0} \{k\}$.

Definition 5.5. Let $\mathbf{I} = (I, I_0, \varepsilon, d)$ and $\mathbf{I}' = (I', I'_0, \varepsilon', d')$ be two seeds with covers, and $k \in I_0$. A *tropical mutation in the direction k* is an involution $\mu_k : \mathbf{I} \rightarrow \mathbf{I}'$ satisfying the following conditions:

- (i) $\mu_k(I_0(i)) = I'_0(i)$;
- (ii) $d'_{\mu_k(i)} = d_i$;
- (iii)

$$\varepsilon'_{\mu_k(i)\mu_k(j)} = \begin{cases} -\varepsilon_{ij} & \text{if } i = k \text{ or } j = k; \\ \varepsilon_{ij} & \text{if } i, j \in I_0(k) \setminus \{k\}; \\ \varepsilon_{ij} - \varepsilon_{ik}b_{kj} & \text{otherwise.} \end{cases}$$

Tropical mutations induce maps between the corresponding seed \mathcal{X} -tori, which are denoted by the same symbols μ_k and given by

$$(5.2) \quad x_{\mu_k(i)} = \begin{cases} x_k^{-1} & \text{if } i = k; \\ x_i x_k^{b_{ki}} & \text{if } i \in I_0(k) \setminus \{k\}; \\ x_i & \text{otherwise.} \end{cases}$$

Remark 5.6. It is easy to see that, as mutations, tropical mutations are involutions.

Remark 5.7. Here is the reason underlying the terminology for *tropical mutation*. Following [FZ02, Example 5.6], let us consider the abelian group (written multiplicatively) freely generated by the cluster variables x_i ($i \in I$), given with the addition

$$\prod_i x_i^{a_i} \boxplus \prod_i x_i^{b_i} := \prod_i x_i^{\min(a_i, b_i)}.$$

This tropical addition \boxplus has not to be confused with the tropical addition \oplus in the usual tropical setting: $a \odot b = a + b$ and $a \oplus b = \min(a, b)$. But is easy to see that they are strongly related: $\prod_i x_i^{a_i} \boxplus \prod_i x_i^{b_i} = \prod_i x_i^{a_i \oplus b_i}$.

It turns out that the left and right tropical mutations, associated respectively to left and right τ -moves $\mathfrak{L}_{\bar{i}}$ and \mathfrak{R}_j (but not \mathfrak{L}_i and $\mathfrak{R}_{\bar{j}}$!), with $i, j \in [1, l]$ and described in the next subsection, can alternatively be defined by the following formula, obtained by tropicalizing the mutation formula in Definition 2.8:

$$(5.3) \quad x_{\mu_k(i)} = \begin{cases} x_k^{-1} & \text{if } i = k; \\ x_i x_k^{(b_{ik})_+} (1 \boxplus x_k)^{-b_{ik}} & \text{otherwise.} \end{cases}$$

The formula (5.3) has relatives in the cluster algebra literature:

- The formula covers the mutation combinatorics [FZ07] attached to a labeled Y -seed (\mathbf{y}, B) defined by $\mathbf{y} = \{x_i | i \in I_0\}$ and $B = (b_{ij})_{i,j \in I_0}$.
- Up to the rescaling $\varepsilon \rightarrow b$ given by (5.1) and the choice of direction, the formula (5.3) is equal to the monomial part μ'_k of mutations respecting to the decomposition given in [FG07b, Section 2.3].
- Compositions of mutations and tropical mutations can be used to describe the modified octohedron recurrence of Henriques and Kamnitzer [HK06]. The related Poisson dynamics on double Bruhat cells involves twist maps and will be given in a separated paper [Bbis].

Remark 5.8. Like the amalgamation product, tropical mutations act on cluster variables x_j associated to outlets, therefore the commutation between them is not always satisfied. In fact a tropical mutation, acting non trivially on a set of cluster variables $\{x_j\}_{j \in J}$ associated to a set of outlets J , commutes with the amalgamation product if and only we have $L \cap J = \emptyset$, where the set L denotes the set of amalgamation, as given in Definition 2.10.

Finally, in the same way that mutations and symmetries were respecting the set of outlets in the definitions of Subsection 2.3, we suppose, from now on, that they respect covers on seeds.

Definition 5.9. A *generalized cluster transformation* linking two seeds (and two seed \mathcal{X} -tori) is a composition of symmetries, mutations, and tropical mutations.

5.1.3. Left and right tropical mutations. We are now ready to describe the way to relate left and τ -moves to particular tropical mutations, respectively called left and right tropical mutations.

Definition 5.10. Let \mathbf{i} be a double word and recall the subsets $I_0^{\mathfrak{L}}(\mathbf{i})$ and $I_0^{\mathfrak{R}}(\mathbf{i})$ of $I_0(\mathbf{i})$ defined in Subsection 3.2.2. From now on, we will denote $\mathbf{I}(\mathbf{i})$ the seed $(I(\mathbf{i}), I_0(\mathbf{i}), \varepsilon(\mathbf{i}), d(\mathbf{i}))$ given with the cover

$$I_0(\mathbf{i}) = I_0^{\mathfrak{L}}(\mathbf{i}) \cup I_0^{\mathfrak{R}}(\mathbf{i}) .$$

Proposition 5.11. *The following tropical mutations are Poisson birational isomorphisms for every double word $\mathbf{i} = i_1 \dots i_n$.*

$$(5.4) \quad \mu_{\binom{i_1}{0}} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathfrak{L}(\mathbf{i})} \quad \text{and} \quad \mu_{\binom{i_n}{N^{i_n}(\mathbf{i})}} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathfrak{R}(\mathbf{i})} .$$

Proof. The birational part is clear, so we focus on the Poisson part. Let us notice that, because right (resp. left) tropical mutations commute with an amalgamated product done on the left side (resp. right side), as detailed in Remark 5.13, it suffices to show that the proposition is true for $\mathbf{i} \in \{i, \bar{i}\}$. Consider the case $\mathbf{i} = i$. So we have $\mathfrak{L}(\mathbf{i}) = \mathfrak{R}(\mathbf{i}) = \bar{i}$. and the equality $b(\mathbf{i}) = 2\varepsilon(\mathbf{i})$. It is then straightforward to check that the matrices $\varepsilon(\mathfrak{L}(\mathbf{i}))$ and $\varepsilon(\mathfrak{R}(\mathbf{i}))$ are equal to $\varepsilon(\bar{i})$. The case $\mathbf{i} = \bar{i}$ is proved in the same way. \square

Definition 5.12. Let $\mathbf{i} = i_1 \dots i_n$ be a double word. The tropical mutations given by equation (5.4) are respectively called *left* and *right tropical mutations* and we denote the associated directions respectively by $\diamond_{i_1}^{\mathcal{L}}$ and $\diamond_{i_n}^{\mathcal{R}}$:

$$\diamond_{i_1}^{\mathcal{L}} = \begin{pmatrix} i_1 \\ 0 \end{pmatrix} \quad \text{and} \quad \diamond_{i_n}^{\mathcal{R}} = \begin{pmatrix} i_n \\ N^{i_n}(\mathbf{i}) \end{pmatrix}.$$

Remark 5.13. Remark 5.8 can be refined in the following way: tropical mutation in a left (resp. right) outlet direction applied to a seed \mathcal{X} -torus $\mathcal{X}_{\mathbf{I}}$ (resp. $\mathcal{X}_{\mathbf{J}}$) is equal to the tropical mutation in the same direction applied to the amalgamated seed \mathcal{X} -torus $\mathcal{X}_{\mathbf{m}(\mathbf{I}, \mathbf{J})}$.

Left and right tropical mutations are easily described on Dynkin quivers. Let us take a Dynkin quiver Γ and let $\mathbf{i} = i_1 \dots i_n$ be the associated double word, so we have $\Gamma = \Gamma_{\mathbf{g}}(\mathbf{i})$. In particular, Γ is obtained by the amalgamation of the elementary Dynkin quivers $\Gamma_{\mathbf{g}}(i_1), \dots, \Gamma_{\mathbf{g}}(i_n)$. Let us remember the behavior between tropical mutations and the amalgamated product given by Remark 5.13; and change the orientation of the arrows of $\Gamma_{\mathbf{g}}(i_1)$ (resp. $\Gamma_{\mathbf{g}}(i_m)$) if we have a left (resp. right) tropical mutation. We get the elementary Dynkin quiver $\Gamma_{\mathbf{g}}(\overline{i_1})$, (resp. $\Gamma_{\mathbf{g}}(\overline{i_m})$). Now, let us perform the amalgamation

$$\mathbf{m} : \Gamma_{\mathbf{g}}(\overline{i_1}) \times \dots \times \Gamma_{\mathbf{g}}(i_n) \rightarrow \Gamma_{\mathbf{g}}(\mathcal{L}(\mathbf{i})) \quad (\text{resp.} \quad \mathbf{m} : \Gamma_{\mathbf{g}}(i_1) \times \dots \times \Gamma_{\mathbf{g}}(\overline{i_n}) \rightarrow \Gamma_{\mathbf{g}}(\mathcal{R}(\mathbf{i}))).$$

The resulting quiver Γ' is therefore the image of Γ by the left (resp. right) tropical mutation. This procedure is illustrated in Figure 13.

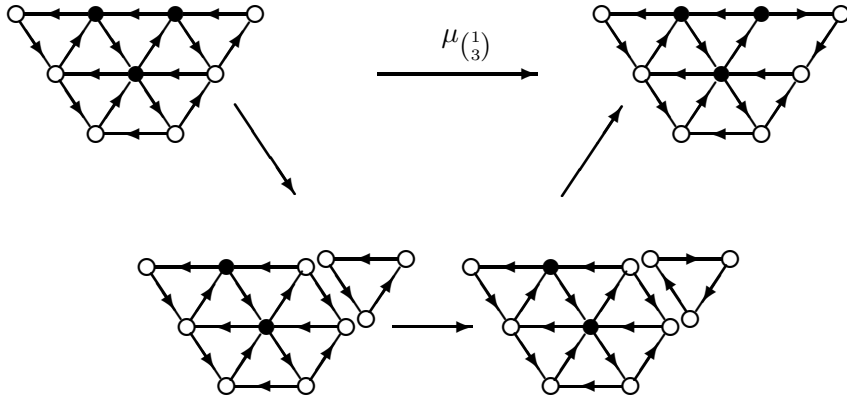


FIGURE 13. The tropical mutation $\mu_{\binom{1}{3}} : \Gamma_{A_3}(123121) \mapsto \Gamma_{A_3}(12312\overline{1})$

5.2. Symmetries on seed \mathcal{X} -tori. In this subsection and in the next one, we define involutions on double words, and the related symmetries on seed \mathcal{X} -tori. They will be use to describe various automorphisms and anti-automorphisms on the group G . Let us recall that the *fundamental weights* $\omega_i \in \mathfrak{h}^*$, given by $\omega_i(h_j) = \delta_{ij}$ for every $i, j \in [1, l]$, are permuted by the transformation $(-w_0)$. We denote $i \mapsto i^*$ the induced permutation of the indices of these weights, that is $\omega_{i^*} = -w_0(\omega_i)$. This automorphism on the Dynkin diagram leads to symmetries on seed \mathcal{X} -tori described by the use of Dynkin quivers: for every $u, v \in W$ and $\mathbf{i} \in R(u, v)$ we define the involutions

$$\begin{array}{ccc} \star : R(u, v) & \rightarrow & R(v^*, u^*) \\ \mathbf{i} & \mapsto & \mathbf{i}^* \end{array} \quad \text{op} : \begin{array}{ccc} R(u, v) & \rightarrow & R(u^{-1}, v^{-1}) \\ \mathbf{i} & \mapsto & \mathbf{i}^{\text{op}} \end{array}$$

in the following way: the double reduced word $\mathbf{i}^\star \in R(v^\star, u^\star)$ is obtained by transforming each letter i of $[1, l] \cup [\bar{1}, \bar{l}]$ into \bar{i}^\star , so if $\mathbf{i} = i_1 \dots i_n$ then $\mathbf{i}^\star = \bar{i}_1^\star \dots \bar{i}_n^\star$, and the double reduced word $\mathbf{i}^{\text{op}} \in R(u^{-1}, v^{-1})$ is obtained by reading \mathbf{i} backwards: $\mathbf{i}^{\text{op}} = i_n \dots i_1$. Now, let

$$\circlearrowleft: R(u, v) \rightarrow R(v^{\star-1}, u^{\star-1})$$

be the map such that the double reduced word $\mathbf{i}^\circlearrowleft \in R(v^{\star-1}, u^{\star-1})$ is obtained by converting each i into \bar{i}^\star and then read the result backwards. Stated otherwise, this transformation is defined by the equality $\circlearrowleft = \text{op} \circ \star$. It induces the following symmetry on seed \mathcal{X} -tori.

$$(5.5) \quad \begin{aligned} \circlearrowleft: \mathcal{X}_{\mathbf{i}} &\longrightarrow \mathcal{X}_{\mathbf{i}^\circlearrowleft} \\ x_{(j)}^{(i)} &\longmapsto x_{(N^{i^\star(\mathbf{i}^\circlearrowleft)-j})} \end{aligned} .$$

Remark 5.14. The involutions \star and op were already defined on reduced words $\mathbf{i} \in R(w_0)$ in [BZ01, Equation (3.1)]. Moreover, the notation \circlearrowleft for the last involution on double reduced words symbolizes a rotation of 180 degrees performed on the quiver $\Gamma_{\mathbf{g}}(\mathbf{i})$, as illustrated in Figure 15.

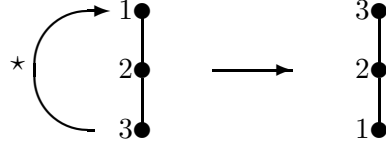


FIGURE 14. The \star involution on Γ_{A_3}

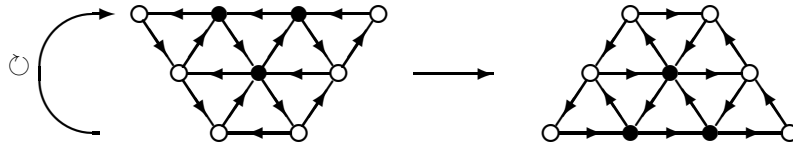


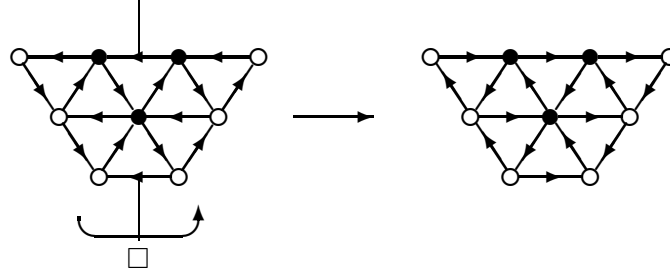
FIGURE 15. The involution $\circlearrowleft: \Gamma_{A_3}(123121) \mapsto \Gamma_{A_3}(\overline{323123})$

In the same way, to any double word \mathbf{i} , we associate the double word \mathbf{i}^\square obtained by reading \mathbf{i} backwards and applying the involution $i \mapsto \bar{i}$ on every letter of the result. In particular, for every $u, v \in W$ we get an involution

$$\square: R(u, v) \rightarrow R(v^{-1}, u^{-1}) .$$

This involution first induces a symmetry on the related seed: it corresponds to a rotation along the vertical axis passing by the center of the Dynkin quiver $\Gamma_{\mathbf{g}}(\mathbf{i})$, as illustrated in Figure 16. It also induces a symmetry on seed \mathcal{X} -tori, also denoted \square and given by:

$$\begin{aligned} \square: \mathcal{X}_{\mathbf{i}} &\longrightarrow \mathcal{X}_{\mathbf{i}^\square} \\ x_{(j)}^{(i)} &\longmapsto x_{(N^{\bar{i}(\mathbf{i})}-j)} \end{aligned}$$

FIGURE 16. The involution $\square : \Gamma_{A_3}(123121) \mapsto \Gamma_{A_3}(\overline{121321})$

Here are now some related isomorphisms on (G, π_G) . Starting with an elementary double word $\mathbf{i} \in \{\mathbf{1}, i, \bar{i}\}$, where $i \in [1, l]$, and then applying the properties of the amalgamation product, we easily prove the following results.

Lemma 5.15. *Let $u, v \in W$ and $\mathbf{i} \in R(u, v)$. For every cluster $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$, let $\mathbf{x}^* \in \mathcal{X}_{\mathbf{i}^*}$ be the cluster such that the following equality is satisfied.*

$$\widehat{w_0} \text{ev}_{\mathbf{i}}(\mathbf{x}) \widehat{w_0}^{-1} = \text{ev}_{\mathbf{i}^*}(\mathbf{x}^*) .$$

Then we have

$$(5.6) \quad x_{(j)}^* = \begin{cases} -x_{(i^*)}^{-1} & \text{if } 0 = j \neq N^{i^*}(\mathbf{i}) \text{ or } 0 \neq j = N^{i^*}(\mathbf{i}); \\ x_{(j)}^{-1} & \text{otherwise.} \end{cases}$$

Lemma 5.16. *Let $u, v \in W$ and $\mathbf{i} \in R(u, v)$. For every cluster $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$, let $\mathbf{x}^{\text{op}} \in \mathcal{X}_{\mathbf{i}^{\text{op}}}$ be the cluster such that the following equality is satisfied.*

$$\text{ev}_{\mathbf{i}}(\mathbf{x})^{-1} = \text{ev}_{\mathbf{i}^{\text{op}}}(\mathbf{x}^{\text{op}}) .$$

Then we have

$$x_{(j)}^{\text{op}} = \begin{cases} -x_{(N^i(\mathbf{i})-j)}^{-1} & \text{if } 0 = j \neq N^i(\mathbf{i}) \text{ or } 0 \neq j = N^i(\mathbf{i}); \\ x_{(N^i(\mathbf{i})-j)}^{-1} & \text{otherwise.} \end{cases}$$

Let us denote $\kappa : G \times G \rightarrow G$ the composition of the conjugacy map associated to the first argument $g \in G$ and the inverse map $x \mapsto x^{-1}$ for the second argument $x \in G$, that is:

$$(5.7) \quad \kappa : (g, x) \mapsto gx^{-1}g^{-1} .$$

Proposition 5.17. *For every $u, v \in W$, $\mathbf{i} \in R(u, v)$, and every $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$, the following equality is satisfied:*

$$\kappa(\widehat{w_0}, \text{ev}_{\mathbf{i}}(\mathbf{x})) = \text{ev}_{\mathbf{i}^{\circ}}(\mathbf{x}^{\circ}) .$$

Proof. The involutions defined by Lemma 5.15, Lemma 5.16 and equation (5.5) on seed \mathcal{X} -tori clearly imply that the relation $\circ = \text{op} \circ \star$ remains valid on seed \mathcal{X} -tori. Therefore Lemma 5.15 and Lemma 5.16 lead to the researched equality. \square

5.3. Chiral dual and other involutions. We introduce other involutions on double words. These ones will be directly useful to describe the combinatorics associated to the twist maps.

Definition 5.18. [FG07b] The *chiral dual* of a seed $\mathbf{I} = (I, I_0, \varepsilon, d)$ is the seed $\mathbf{I}^\circ = (I, I_0, -\varepsilon, d)$. Chiral dual induces an involutive map between the corresponding seed \mathcal{X} -tori, which is denoted in the same way and given by $x_{\circ(i)} = x_i^{-1}$.

Remark 5.19. It can be checked that the chiral dual commutes with cluster transformations but not with tropical mutations.

The chiral dual leads to the following involution on double reduced words, also denoted \circ . For every $u, v \in W$ and every double word $\mathbf{i} \in D(u, v)$, let $\mathbf{i}^\circ \in D(v, u)$ be the double word obtained by applying the map $i \mapsto \bar{i}$ as a homomorphism on the double word \mathbf{i} . It is clear that the double word \mathbf{i}° is reduced if and only if \mathbf{i} is reduced. The following result is immediate.

Lemma 5.20. *For every double word \mathbf{i} , the chiral dual $\mathbf{I}(\mathbf{i})^\circ$ of the seed $\mathbf{I}(\mathbf{i})$ associated to the double word \mathbf{i} is the seed $\mathbf{I}(\mathbf{i}^\circ)$ associated to the double word \mathbf{i}° .*

Here is now the link with the group G . Let us recall that the *involutive Cartan group automorphism* $\theta : G \rightarrow G : x \mapsto x^\theta$ is the map given by

$$(5.8) \quad a^\theta = a^{-1} \in H, \quad E^{i^\theta} = F^i, \quad F^{i^\theta} = E^i.$$

Proposition 5.21. *For every $u, v \in W$ and every double reduced word $\mathbf{i} \in R(u, v)$, the following diagram commutes. Its vertical edges are labeled by birational Poisson isomorphisms whereas its horizontal edges are labeled by birational anti-Poisson isomorphisms.*

$$\begin{array}{ccc} & \circ & \\ \mathcal{X}_{\mathbf{i}} & \xrightarrow{\quad} & \mathcal{X}_{\mathbf{i}^\circ} \\ \text{ev}_{\mathbf{i}} \downarrow & & \downarrow \text{ev}_{\mathbf{i}^\circ} \\ (G^{u,v}, \pi_G) & \xrightarrow{\quad \theta \quad} & (G^{v,u}, \pi_G) \end{array}$$

Proof. Because the involution \circ commutes with the amalgamated product, which intertwines the product on G via the evaluation map by equation (3.5), and because θ is a group automorphism for this product on G , we just have to focus on the case $\mathbf{i} \in \{i, \bar{i}\}$. The result is then easily derived from the definition of the involutions \circ and θ , and the formula (3.3). \square

5.4. Generalized cluster transformations and twist maps on (G, π_G) . In this subsection, we use generalized cluster transformations to give the cluster combinatorics underlying the Fomin-Zelevinsky twist maps.

5.4.1. Tropical mutations and twist maps. To $i \in [1, l] \cup [\bar{1}, \bar{l}]$, we associated the positive letter $|i| \in [1, l]$ given by the formula

$$(5.9) \quad |i| = \begin{cases} i & \text{if } i \in [1, l] ; \\ \bar{i} & \text{otherwise .} \end{cases}$$

Let $w, w' \in W$. We denote $w \rightarrow w'$ if and only if we can find a letter $i \in [1, l]$ such that $w = s_i w'$ and $\ell(w) = \ell(w') + 1$, and denote \leq the *right weak order* on W , i.e. $w' \leq w$ if there exists a chain $w \rightarrow \dots \rightarrow w'$.

Let us recall that for every $w' \leq w$, a reduced word $\mathbf{i} = i_1 \dots i_{\ell(w)} \in R(w)$ is said to be *adapted to w'* if we have the equality $s_{i_1} \dots s_{i_{\ell(w')}} = w'$. We extend this notation to every double word $\mathbf{i} = i_1 \dots i_n$ by setting $|\mathbf{i}| := |i_1| \dots |i_n|$.

Definition 5.22. Let $e < w_1 \leq u, e < w_2 \leq v \in W$. A double reduced word $\mathbf{i} = i_1 \dots i_n \in R(u, v)$ is said to be \mathfrak{L} -*adapted to w_1* (resp. \mathfrak{R} -*adapted to w_2*) if the reduced word

$$|i_1 \dots i_{\ell(w_1)}| \in R(w_1) \quad (\text{resp.} \quad |i_{\ell(u)+\ell(w_2^{-1}v)+1} \dots i_{\ell(u)+\ell(v)}| \in R(w_2))$$

is adapted to w_1 (resp. adapted to w_2). And the double word \mathbf{i} is (w_1, w_2) -*adapted* if it is \mathfrak{L} -adapted to w_1 and \mathfrak{R} -adapted to w_2 . In particular, a (w_1, w_2) -adapted double reduced word is (w'_1, w'_2) -adapted for every $w'_1 \leq w_1, w'_2 \leq w_2$.

For example, the double reduced word $\mathbf{i} = \overline{21}2$ is \mathfrak{L} -adapted for the elements $s_2, s_2 s_1 \in W$, \mathfrak{R} -adapted for $s_2, (s_2 s_1, s_2)$ -adapted, and (s_2, s_2) -adapted; whereas the double reduced word $\mathbf{j} = 2\overline{21}$ is neither \mathfrak{L} -adapted, nor \mathfrak{R} -adapted, hence nor (w_1, w_2) -adapted, for any $w_1, w_2 \in W$.

Remark 5.23. For every $u, v \in W$, if the double reduced word $\mathbf{i} \in R(u, v)$ is (u, v) -adapted, then its first $\ell(u)$ letters (resp. $\ell(v)$ last letters) are negative (resp. positive) and give a reduced expression for u (resp. v). Moreover, the following assertions are equivalent for every double reduced word $\mathbf{i} \in R(u, v)$:

- the double reduced word \mathbf{i} is (u, v) -adapted;
- the double reduced word \mathbf{i} is \mathfrak{L} -adapted to u ;
- the double reduced word \mathbf{i} is \mathfrak{R} -adapted to v .

Proposition 5.24. *The following equalities are satisfied for every $u, v \in W$, every (u, v) -adapted double word $\mathbf{i} = i_1 \dots i_n \in R(u, v)$, and every $\mathbf{x} \in \mathcal{X}_1$.*

$$\begin{aligned} [\text{ev}_{\mathbf{i}}(\mathbf{x}) \widehat{v^{-1}}]_{\leq 0} &= [\text{ev}_{\mathfrak{R}(\mathbf{i})} \circ \mu_{\binom{i_n}{N^{i_n}(\mathbf{i})}}(\mathbf{x}) \widehat{s_{i_n} v^{-1}}]_{\leq 0} \\ [\widehat{u^{-1}} \text{ev}_{\mathbf{i}}(\mathbf{x})]_{\geq 0} &= [\widehat{s_{i_1} u^{-1}} \text{ev}_{\mathfrak{L}(\mathbf{i})} \circ \mu_{\binom{i_1}{0}}(\mathbf{x})]_{\geq 0}. \end{aligned}$$

Proof. Let us remember the map $\varphi_j : \text{SL}(2, \mathbb{C}) \hookrightarrow G$ defined in Subsection 2.1. For any nonzero $t \in \mathbb{C}$ and any $i \in [1, l]$, let us denote

$$(5.10) \quad x_i(t) = \varphi_i \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad x_{\bar{i}}(t) = \varphi_i \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

For every $j \in [1, l]$, the following equality, easily checked on $\text{SL}(2, \mathbb{C})$ by elementary matrix calculus, can be extended on G using the map φ_j of Subsection 2.1.

$$(5.11) \quad \widehat{s_j^{-1}} x_{\bar{j}}(t) = x_{\bar{j}}(-t^{-1}) t^{h_j} x_j(t^{-1}).$$

Using the definition of tropical mutation, and the fact that tropical mutations commute with the amalgamated product according to Remark 5.13, we deduce:

$$(5.12) \quad \widehat{s_{i_1}^{-1}} \text{ev}_{\mathbf{i}}(\mathbf{x}) = x_{\bar{i}_1}^{-1}(-x_{\binom{i_1}{0}}^{-1}) \text{ev}_{\mathfrak{L}(\mathbf{i})} \circ \mu_{\binom{i_1}{0}}(\mathbf{x}).$$

Moreover, we have the inequality $\ell(s_{i_1} u) < \ell(u)$ because the double word \mathbf{i} is \mathfrak{L} -adapted to u . This inequality implies that $\widehat{s_{i_1} u^{-1}} x_{\bar{j}}(t) \widehat{s_{i_1} u} \in N_-$ for every $t \in \mathbb{C}$. Therefore the

second equation is proved. The first equation is proved in the same way, using the following equality instead of (5.11).

$$(5.13) \quad x_i(t)\widehat{s}_i = x_{\bar{i}}(t^{-1})t^{h_i}x_i(-t^{-1}) .$$

□

Definition 5.25. [FZ99, Definition 1.5] Let $u, v \in W$. The *twist map* $\zeta_\theta^{u,v} : x \mapsto x'$ is the map defined by

$$(5.14) \quad x' = ([\widehat{u}^{-1}x]_{-}^{-1}\widehat{u}^{-1}xv^{-1}[\widehat{xv^{-1}}]_{+}^{-1})^\theta .$$

Because of [FZ99, Theorem 1.6], the right side of (5.14) is well defined for every $x \in G^{u,v}$ and the twist map $\zeta_\theta^{u,v}$ establishes a biregular isomorphism between $G^{u,v}$ and $G^{u^{-1},v^{-1}}$. Let us define the related map

$$(5.15) \quad \begin{array}{ccc} \zeta^{u,v} : G^{u,v} & \longrightarrow & G^{v^{-1},u^{-1}} \\ x & \longmapsto & [\widehat{u}^{-1}x]_{-}^{-1}\widehat{u}^{-1}xv^{-1}[\widehat{xv^{-1}}]_{+}^{-1} . \end{array}$$

So we get the equality $\zeta_\theta^{u,v} = \theta \circ \zeta^{u,v}$. Let us notice that for every $x \in G^{u,1}$ and $y \in G^{1,v}$ we have the relations $\zeta^{u,1}(x) = [\widehat{u}^{-1}x]_{\geq 0}$ and $\zeta^{1,v}(y) = [y\widehat{v}]_{\leq 0}$.

Remark 5.26. Because the map (5.15) will appear a lot in what follow, it will be useful to denote it also by the expression "twist map". When we will need to avoid confusion, we will refer to the map (5.14) as the "Fomin-Zelevinsky twist map".

5.4.2. Generalized cluster transformations and twist maps on (B_\pm, π_G) . We are going to describe twist maps at the level of seed \mathcal{X} -tori. Let us start to associate a generalized cluster transformation to any twist map on (B_\pm, π_G) . To do that, we first need to sharpen the preceding involution \square on double words. For every positive reduced word $\mathbf{i} = i_1 \dots i_n$, every negative reduced word $\mathbf{j} = j_1 \dots j_n$, and every $k \in [1, n+1]$, we introduce the double words

$$(5.16) \quad \begin{array}{l} \mathbf{i}(k) = \mathbf{i}(k)_- \mathbf{i}(k)_+ \quad \text{where} \quad \mathbf{i}(k)_+ = i_1 \dots i_{k-1} \quad \text{and} \quad \mathbf{i}(k)_- = \overline{i_n} \dots \overline{i_k} \\ \mathbf{j}(k) = \mathbf{j}(k)_- \mathbf{j}(k)_+ \quad \text{where} \quad \mathbf{j}(k)_+ = j_{k-1} \dots j_1 \quad \text{and} \quad \mathbf{j}(k)_- = \overline{j_k} \dots \overline{j_n} . \end{array}$$

The following insight on these double reduced words will be developed in the next section, by considering the W -permutohedron. It is derived from an easy induction on the number $k \in [1, \ell(w)]$ that appear in the statement.

Lemma 5.27. Let $w \in W$, $\mathbf{i} = i_1 \dots i_{\ell(w)} \in R(1, w)$ be a positive reduced word, and $w_{\geq k} = s_{i_k} \dots s_{i_{\ell(w)}}$ be the element of w associated to any $k \in [1, k]$. The double reduced word $\mathbf{i}(k)$ belongs to the set $R(w_{\geq k}^{-1}, ww_{\geq k}^{-1})$.

Remark 5.28. The involution \square on positive or negative words is rediscovered from equation (5.16) because of the following equalities $\mathbf{i} = \mathbf{i}(n+1)$, $\mathbf{j} = \mathbf{j}(1)$ and $\mathbf{i}^\square = \mathbf{i}(1)$, $\mathbf{j}^\square = \mathbf{j}(n)$.

Example 5.29. Let us choose the positive reduced word $\mathbf{i} = 121 \in R(1, w_0)$, when $\mathfrak{g} = A_2$. We then get the following double reduced words $\mathbf{i}(4) = 121$, $\mathbf{i}(3) = \overline{1}12$, $\mathbf{i}(2) = \overline{1}2\overline{1}$, and $\mathbf{i}(1) = \overline{1}2\overline{1}$. In the same way, if we consider the negative double word $\mathbf{j} = \overline{1}2\overline{1} \in R(w_0, 1)$, we get the following double reduced words $\mathbf{j}(1) = \overline{1}2\overline{1}$, $\mathbf{j}(2) = \overline{1}2\overline{1}$, $\mathbf{j}(3) = \overline{1}12$ and $\mathbf{j}(4) = 121$.

For every positive reduced word $\mathbf{i} = i_1 \dots i_n$, every negative reduced word $\mathbf{j} = j_1 \dots j_n$ and every $k \in [1, n]$, we define the generalized cluster transformations $\zeta_{\mathbf{i}(k)} : \mathcal{X}_{\mathbf{i}(k)} \rightarrow \mathcal{X}_{\mathbf{i}(k-1)}$ and $\zeta_{\mathbf{j}(k)} : \mathcal{X}_{\mathbf{j}(k)} \rightarrow \mathcal{X}_{\mathbf{j}(k-1)}$ by the following formulas

$$(5.17) \quad \begin{aligned} \zeta_{\mathbf{i}(k)} &= \mu_{\binom{i_k}{N^{i_k}(\mathbf{i}(k)_-)}} \circ \mu_{\binom{i_k}{N^{i_k}(\mathbf{i}(k)_-)+1}} \circ \dots \circ \mu_{\binom{i_k}{N^{i_k}(\mathbf{i})}} \\ \text{and} \\ \zeta_{\mathbf{j}(k)} &= \mu_{\binom{j_k}{0}} \circ \mu_{\binom{j_k}{1}} \circ \dots \circ \mu_{\binom{j_k}{N^{j_k}(\mathbf{j}(k)_-)-1}}. \end{aligned}$$

Example 5.30. Here are the generalized cluster transformations related to the double reduced words of the previous example.

$$\begin{aligned} \zeta_{\mathbf{i}(4)} &= \mu_{\binom{1}{1}} \circ \mu_{\binom{1}{2}}, \quad \zeta_{\mathbf{i}(3)} = \mu_{\binom{2}{1}}, \quad \zeta_{\mathbf{i}(2)} = \mu_{\binom{1}{2}}, \\ \text{and} \\ \zeta_{\mathbf{j}(1)} &= \mu_{\binom{1}{1}} \circ \mu_{\binom{1}{0}}, \quad \zeta_{\mathbf{j}(2)} = \mu_{\binom{2}{0}}, \quad \zeta_{\mathbf{j}(3)} = \mu_{\binom{1}{0}}. \end{aligned}$$

Corollary 5.31. *For every $u, v \in W$, $\mathbf{i} \in R(1, v)$ and $\mathbf{j} \in R(u, 1)$, the following maps are Poisson birational isomorphisms.*

$$(5.18) \quad \begin{aligned} \zeta_{\mathbf{i}(\geq k)} : \mathcal{X}_{\mathbf{i}} &\longrightarrow \mathcal{X}_{\mathbf{i}(k-1)} & \zeta_{\mathbf{j}(\leq k)} : \mathcal{X}_{\mathbf{j}} &\longrightarrow \mathcal{X}_{\mathbf{j}(k-1)} \\ \mathbf{x} &\longmapsto \zeta_{\mathbf{i}(k)} \circ \dots \circ \zeta_{\mathbf{i}(n)}(\mathbf{x}) & \mathbf{x} &\longmapsto \zeta_{\mathbf{j}(k)} \circ \dots \circ \zeta_{\mathbf{j}(1)}(\mathbf{x}). \end{aligned}$$

Proof. We use Proposition 5.11, equation (5.17), and the fact that mutations are Poisson birational isomorphisms. \square

Remark 5.32. The generalized cluster transformations (5.18) will be used in Section 9 to describe the unipotent parts of the dual Poisson-Lie group (G^*, π_{G^*}) .

Example 5.33. Let us keep the positive reduced word $\mathbf{i} = 121 \in R(1, w_0)$, when $\mathfrak{g} = A_2$. The generalized cluster transformations associated to the double reduced words of the example above are then the following. (We denote $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$ the cluster $(x_{\binom{1}{0}}, x_{\binom{1}{1}}, x_{\binom{1}{2}}, x_{\binom{2}{0}}, x_{\binom{2}{1}})$.)

$$\begin{aligned} \zeta_{\mathbf{i}(\geq 3)}(\mathbf{x}) &= \zeta_{\mathbf{i}(3)}(\mathbf{x}) = \begin{pmatrix} x_{\binom{1}{0}}(1 + x_{\binom{1}{1}}), & x_{\binom{1}{1}}^{-1}, & x_{\binom{1}{2}}^{-1}(1 + x_{\binom{1}{1}}), \\ x_{\binom{2}{0}}(1 + x_{\binom{1}{1}}^{-1})^{-1}, & x_{\binom{2}{1}}x_{\binom{1}{2}} \end{pmatrix}; \\ \zeta_{\mathbf{i}(\geq 2)}(\mathbf{x}) &= \zeta_{\mathbf{i}(2)} \circ \zeta_{\mathbf{i}(3)}(\mathbf{x}) = \begin{pmatrix} x_{\binom{1}{0}}(1 + x_{\binom{1}{1}}), & x_{\binom{1}{1}}^{-1}, & x_{\binom{2}{1}}(1 + x_{\binom{1}{1}}), \\ x_{\binom{2}{0}}(1 + x_{\binom{1}{1}}^{-1})^{-1}, & x_{\binom{2}{1}}^{-1}x_{\binom{1}{2}}^{-1} \end{pmatrix}; \\ \zeta_{\mathbf{i}(\geq 1)}(\mathbf{x}) &= \zeta_{\mathbf{i}(1)} \circ \zeta_{\mathbf{i}(2)} \circ \zeta_{\mathbf{i}(3)}(\mathbf{x}) = \begin{pmatrix} x_{\binom{1}{0}}(1 + x_{\binom{1}{1}}), & x_{\binom{1}{1}}^{-1}, & x_{\binom{2}{1}}^{-1}(1 + x_{\binom{1}{1}})^{-1}, \\ x_{\binom{2}{0}}(1 + x_{\binom{1}{1}}^{-1})^{-1}, & x_{\binom{1}{2}}^{-1}(1 + x_{\binom{1}{1}}) \end{pmatrix}. \end{aligned}$$

Special cases of the generalized cluster transformations (5.18) are given by the following birational Poisson isomorphisms. These are the ones that we are going to associate to twist maps on (G, π_G) .

$$(5.19) \quad \zeta_{\mathbf{i}} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{i}\square} : \mathbf{x} \mapsto \zeta_{\mathbf{i}(\geq 1)}(\mathbf{x}) \quad \text{and} \quad \zeta_{\mathbf{j}} : \mathcal{X}_{\mathbf{j}} \rightarrow \mathcal{X}_{\mathbf{j}\square} : \mathbf{x} \mapsto \zeta_{\mathbf{j}(\leq \ell(u))}(\mathbf{x}).$$

Proposition 5.34. *Let $w' \leq u, w \leq v \in W$, $\mathbf{i} \in R(w'^{-1}u, w'^{-1})$ and $\mathbf{j} \in R(w^{-1}, vw^{-1})$ be double reduced words respectively $(w'^{-1}u, w'^{-1})$ -adapted and (w^{-1}, vw^{-1}) -adapted. The following equalities are satisfied for every cluster $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$ and $\mathbf{y} \in \mathcal{X}_{\mathbf{j}}$:*

$$[\widehat{w'^{-1}u}^{-1} \text{ev}_{\mathbf{i}}(\mathbf{x})]_{\geq 0} = \text{ev}_{\mathbf{i} \square \mathbf{i}_+} \circ \zeta_{\mathbf{i}_-}(\mathbf{x}) \quad \text{and} \quad [\text{ev}_{\mathbf{j}}(\mathbf{y}) \widehat{wv^{-1}}]_{\leq 0} = \text{ev}_{\mathbf{j} \square \mathbf{j}_+} \circ \zeta_{\mathbf{j}_+}(\mathbf{y}) .$$

Proof. Let $w \in W$ and $j \in [1, l]$ be such that $w < ws_j \leq v$ for the right weak order. The following equality is satisfied for every $g \in G^{v,1}$.

$$[\widehat{s_j^{-1}}[\widehat{w}^{-1}g]_{\geq 0}]_{\geq 0} = [\widehat{s_j^{-1}}\widehat{w}^{-1}g]_{\geq 0} .$$

An induction on the length of v , involving at each step the equation (5.12), Theorem 3.7 and the definition (5.19) leads to the second equality. The first equality is proved in the same way. \square

Corollary 5.35. *For every $u, v \in W$, every (double) reduced words $\mathbf{i} \in R(1, u)$ and $\mathbf{j} \in R(v, 1)$, and every cluster $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$ and $\mathbf{y} \in \mathcal{X}_{\mathbf{j}}$, the following equalities are satisfied:*

$$(5.20) \quad [\widehat{u}^{-1} \text{ev}_{\mathbf{i}}(\mathbf{x})]_{\geq 0} = \text{ev}_{\mathbf{i} \square} \circ \zeta_{\mathbf{i}}(\mathbf{x}) \quad \text{and} \quad [\text{ev}_{\mathbf{j}}(\mathbf{y}) \widehat{v^{-1}}]_{\leq 0} = \text{ev}_{\mathbf{j} \square} \circ \zeta_{\mathbf{j}}(\mathbf{y}) .$$

Proof. We apply Proposition 5.34 with $w = w' = e$. \square

Corollary 5.36. *The following equality is satisfied for every $v \in W$ and every reduced words $\mathbf{i}, \mathbf{j} \in R(1, v)$, or $\mathbf{i}, \mathbf{j} \in R(v, 1)$*

$$\mu_{\mathbf{i} \square \rightarrow \mathbf{j} \square} \circ \zeta_{\mathbf{i}} = \zeta_{\mathbf{j}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}} .$$

Proof. We suppose that $\mathbf{i}, \mathbf{j} \in R(1, v)$. Let us recall that the involution \square maps double reduced words to double reduced words, and that the evaluation map $\text{ev}_{\mathbf{j}}$ associated to any double reduced word \mathbf{j} is birational because of Theorem 3.4. Therefore an equality $\mathbf{y} = \mathbf{z}$ between cluster variables on $\mathcal{X}_{\mathbf{i} \square}$ is satisfied if and only if the equality $\text{ev}_{\mathbf{i} \square}(\mathbf{y}) = \text{ev}_{\mathbf{i} \square}(\mathbf{z})$ is satisfied on G . Now, it suffices to apply Theorem 3.7 and the second equation of (5.20) to obtain the following equality for every $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$. The case $\mathbf{i}, \mathbf{j} \in R(v, 1)$ is proved in the same way.

$$\text{ev}_{\mathbf{i} \square} \circ \zeta_{\mathbf{i}}(\mathbf{x}) = [\text{ev}_{\mathbf{j}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}}(\mathbf{x}) \widehat{v^{-1}}]_{\leq 0} = \text{ev}_{\mathbf{j} \square} \circ \zeta_{\mathbf{j}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}}(\mathbf{x}) = \text{ev}_{\mathbf{i} \square} \circ \mu_{\mathbf{j} \square \rightarrow \mathbf{i} \square} \circ \zeta_{\mathbf{j}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}}(\mathbf{x}) .$$

\square

5.4.3. From twist maps on (B_{\pm}, π_G) to twist maps on (G, π_G) . We are now ready to give the generalized cluster transformations associated to any twist maps on (G, π_G) . For every double word \mathbf{i} , let \mathbf{i}_+ and \mathbf{i}_- be respectively the words obtained by erasing all the negative and positive letters of \mathbf{i} , without changing the order of the remaining letters. The word \mathbf{i}_+ (resp. \mathbf{i}_-) is called the *positive part* (resp. *negative part*) of \mathbf{i} . In particular, the double word \mathbf{i} is linked by compositions of mixed 2-moves to the double words $\mathbf{i}_+ \mathbf{i}_-$ and $\mathbf{i}_- \mathbf{i}_+$. (This definition is compatible with the notation used in equation (5.16).) Following Corollary 5.36, we then introduce for every $\mathbf{i} \in R(u, v)$ the maps $\zeta_{\mathbf{i}} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{i} \square}$ and $\zeta_{\mathbf{i} \square} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{i} \square}$ by the following formulas and state the main result of this subsection.

$$(5.21) \quad \zeta_{\mathbf{i}} = \mu_{\mathbf{i} \square \rightarrow \mathbf{i} \square} \circ \zeta_{\mathbf{i}_-} \circ \zeta_{\mathbf{i}_+} \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}_+ \mathbf{i}_-} \quad \text{and} \quad \zeta_{\mathbf{i} \square} = \square \circ \zeta_{\mathbf{i}} .$$

Theorem 5.37. *For every $u, v \in W$ and every double reduced word $\mathbf{i} \in R(u, v)$, the following diagrams are commutative. All edges of the left diagram are Poisson birational isomorphisms, whereas vertical edges of the right diagram are Poisson birational isomorphisms and horizontal edges are anti-Poisson birational isomorphisms.*

$$\begin{array}{ccc}
 & \xrightarrow{\zeta_{\mathbf{i}}} & \\
 \text{ev}_{\mathbf{i}} \swarrow & & \searrow \text{ev}_{\mathbf{i}\square} \\
 \mathcal{X}_{\mathbf{i}} & & \mathcal{X}_{\mathbf{i}\square} \\
 \downarrow (G^{u,v}, \pi_G) & & \downarrow (G^{v^{-1}, u^{-1}}, \pi_G) \\
 & \xrightarrow{\zeta^{u,v}} & \\
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \xrightarrow{\zeta_{\mathbf{i}\bigcirc}} & \\
 \text{ev}_{\mathbf{i}} \swarrow & & \searrow \text{ev}_{\mathbf{i}\text{op}} \\
 \mathcal{X}_{\mathbf{i}} & & \mathcal{X}_{\mathbf{i}\text{op}} \\
 \downarrow (G^{u,v}, \pi_G) & & \downarrow (G^{u^{-1}, v^{-1}}, \pi_G) \\
 & \xrightarrow{\zeta_{\theta}^{u,v}} &
 \end{array}$$

Proof. Let first notice that Proposition 5.21 and the definition of $\zeta^{u,v}$ and $\zeta_{\theta}^{u,v}$ imply that the commutativity of the right diagram can be derived from the commutativity of the left one. So, let us focus on the left one. The cases $(u, v) = (u, 1)$ and $(u, v) = (1, v)$ are proved by Proposition 5.34. Moreover, among all the remaining cases, it suffices to prove the case $\mathbf{i} = \mathbf{i}_- \mathbf{i}_+$ (with $\mathbf{i}_- \in R(u, 1)$ and $\mathbf{i}_+ \in R(1, v)$), because of the definition (5.21) of $\zeta_{\mathbf{i}}$ and Theorem 3.7. The demonstration relies on the following equality (5.22), borrowed to [GSV03, Theorem 3.1]. For every $x \in G^{u,v}$, the definition (5.15) leads to:

$$\begin{aligned}
 \zeta^{u,v}(x) &= [\widehat{u^{-1}x}]_-^{-1} \widehat{u^{-1}xv^{-1}} [\widehat{xv^{-1}}]_+^{-1} \\
 &= [\widehat{u^{-1}x}]_0 [\widehat{u^{-1}x}]_+ v^{-1} [\widehat{xv^{-1}}]_+^{-1} \\
 (5.22) \quad &= [\widehat{u^{-1}x}]_- [\widehat{u^{-1}x}]_0 [\widehat{u^{-1}x}]_+^{-1} (x^{-1}x) \widehat{v^{-1}} [\widehat{xv^{-1}}]_+^{-1} \\
 &= [\widehat{u^{-1}x}]_{\leq 0} [\widehat{u^{-1}x}]_{\geq 0} [x]_+ x^{-1} [\widehat{xv^{-1}}]_{\leq 0} \\
 &= [\widehat{u^{-1}x}]_{\leq 0} [\widehat{u^{-1}x}]_{\geq 0} [x]_+ x^{-1} [x]_- [\widehat{xv^{-1}}]_{\leq 0} \\
 &= \zeta^{u,1}([x]_{\leq 0}) [x]_0^{-1} \zeta^{1,v}([x]_{\geq 0}) .
 \end{aligned}$$

Let $\mathbf{i} = \mathbf{i}_- \mathbf{i}_+ \in R(u, v)$ and $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$, $\mathbf{x}_- \in \mathcal{X}_{\mathbf{i}_-}$, $\mathbf{x}_+ \in \mathcal{X}_{\mathbf{i}_+}$ be the cluster variables such that the equalities $x = \text{ev}_{\mathbf{i}}(\mathbf{x})$ and $\mathbf{m}(\mathbf{x}_-, \mathbf{x}_+) = \mathbf{x}$ are satisfied. We start by introducing the following maps $\pi_{\mathbf{j}} : \mathcal{X}_{\mathbf{j}} \rightarrow \mathcal{X}_{\mathbf{1}}$, for every double word \mathbf{j} , given by:

$$(5.23) \quad \pi_{\mathbf{j}} : \mathcal{X}_{\mathbf{j}} \rightarrow \mathcal{X}_{\mathbf{1}} \\
 x_{\pi_{\mathbf{i}}(\binom{i}{j})} = x_{\binom{i}{0}} x_{\binom{i}{1}} \dots x_{\binom{i}{N^i(i)}} .$$

We use these maps to define the elements $\mathbf{x}_{\leq 0} \in \mathcal{X}_{\mathbf{i}_-}$, $\mathbf{x}_0 \in \mathcal{X}_{\mathbf{1}}$ and $\mathbf{x}_{\geq 0} \in \mathcal{X}_{\mathbf{i}_+}$ related to \mathbf{x} in the following way. (The associated equalities are easily proved.)

$$(5.24) \quad \mathbf{x}_{\leq 0} = \mathbf{m}(\mathbf{x}_-, \pi_{\mathbf{i}_+}(\mathbf{x}_+)) \quad , \quad \mathbf{x}_0 = \pi_{\mathbf{i}}(\mathbf{x}) \quad \text{and} \quad \mathbf{x}_{\geq 0} = \mathbf{m}(\pi_{\mathbf{i}_-}(\mathbf{x}_-), \mathbf{x}_+) . \\
 [x]_{\leq 0} = \text{ev}_{\mathbf{i}_-}(\mathbf{x}_{\leq 0}) \quad , \quad [x]_0^{-1} = \text{ev}_{\mathbf{1}}(\mathbf{x}_0^{\bigcirc}) \quad \text{and} \quad [x]_{\geq 0} = \text{ev}_{\mathbf{i}_+}(\mathbf{x}_{\geq 0}) .$$

From Remark 5.13 we now get the following relations:

$$\begin{aligned}
 \zeta_{\mathbf{i}_-}(\mathbf{x}_{\leq 0}) &= \mathbf{m}(\zeta_{\mathbf{i}_-}(\mathbf{x}_-), \pi_{\mathbf{i}_+}(\mathbf{x}_+)) \quad \text{and} \quad \zeta_{\mathbf{i}_+}(\mathbf{x}_{\geq 0}) = \mathbf{m}(\pi_{\mathbf{i}_-}(\mathbf{x}_-), \zeta_{\mathbf{i}_+}(\mathbf{x}_+)) ; \\
 \zeta_{\mathbf{i}}(\mathbf{x}) &= \zeta_{\mathbf{i}_-} \circ \zeta_{\mathbf{i}_+}(\mathbf{x}) = \mathbf{m}(\zeta_{\mathbf{i}_-}(\mathbf{x}_{\leq 0}), \mathbf{x}_0^{\bigcirc}, \zeta_{\mathbf{i}_+}(\mathbf{x}_{\geq 0})) = \zeta_{\mathbf{i}_+} \circ \zeta_{\mathbf{i}_-}(\mathbf{x}) .
 \end{aligned}$$

Proposition 5.34 and equalities (5.22), (5.24) then lead to $\zeta^{u,v}(\text{ev}_i(\mathbf{x})) = \text{ev}_{i\Box}(\zeta_i(\mathbf{x}))$. Finally, the Poisson and birational statements are clear from Theorem 3.7, Proposition 5.35 and Proposition 5.21. \square

6. τ -COMBINATORICS, W -PERMUTOHEDRON, AND EVALUATIONS ON (G, π_G)

We will refer to τ -combinatorics as the combinatorics on double reduced words generated by generalized d -moves and enriched with right tropical moves. The idea is to prepare the ground for the cluster combinatorics related to twisted evaluations and dual Poisson-Lie groups, developed in Section 8. Here, we associate a family of cluster \mathcal{X} -varieties to every double Bruhat cell $G^{w,1}$ by linking cluster \mathcal{X} -varieties with tropical mutations via the W -permutohedron associated to the Lie algebra \mathfrak{g} .

6.1. The W -permutohedron, \uparrow -paths and \downarrow -paths. We recall here that any reduced expression of any element of W can be described as a monotone paths on a particular polytope: the W -permutohedron. (or moment polytope, or weight polytope [P05]). Let us recall that Λ denotes the integer weight lattice associated to \mathfrak{g} and denote $\Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R}$ the weight space. The roots in Π span the root lattice $L \subseteq \Lambda$. The associated Weyl group W acts on the weight space $\Lambda_{\mathbb{R}}$. For $x \in \Lambda_{\mathbb{R}}$, we can define the W -permutohedron $P_W(x)$ as the convex hull of a Weyl group orbit:

$$P_W(x) := \text{ConvexHull}(w(x) | w \in W) \subset \Lambda_{\mathbb{R}}.$$

For the Lie type A_n , the W -permutohedron $P_W(x)$ is the permutohedron $P_{n+1}(x)$ defined as the convex hull of all vectors obtained from (x_1, \dots, x_{n+1}) by permutations of the coordinates:

$$P_{n+1}(x_1, \dots, x_{n+1}) := \text{ConvexHull}((x_{w(1)}, \dots, x_{w(n+1)}) | w \in S_{n+1}).$$

From now on, we fix a generic $x \in \Lambda_{\mathbb{R}}$ such that the associated W -permutohedron $P_W(x)$ has maximal dimension. This W -permutohedron will be (abusively) denoted P_W . It is well-known that we can label vertices and edges of P_W respectively by the set of elements of W and the set of elementary reflections $s_i \in S$ that generates W , in such a way that

- every vertex has a different label;
- every labeled vertices w_1 and w_2 of P_W are related by a labeled edge s_i if and only if the equality $w_2 = w_1 s_i$ is satisfied.

In particular, the number of vertices of P_W is given by the cardinal of W . When we draw a picture of the W -permutohedron P_W , the bottom vertex can be associated with the identity element $1 \in W$, so that the top vertex is the longest element $w_0 \in W$. As remarked in [FR07], a reduced word for w then corresponds to a path along edges from 1 to w which moves up in a monotone fashion. Let us call \uparrow -path a path along edges of P_W which moves up in a monotone fashion on P_W . In the same way, a path which moves down in a monotone fashion will be called a \downarrow -path. A \uparrow -path relating the vertex w to the vertex w' is called a $w \nearrow w'$ -path and the corresponding \downarrow -path is called a $w' \searrow w$ -path. In particular, a \uparrow -path along edges from 1 to w is called a \uparrow_w -path. The following result is clear.

Proposition 6.1. *Let $u \leq v \in W$. The $u \nearrow v$ -paths (resp. $v \searrow u$ -paths) are in bijection with reduced expressions of the element $vu^{-1} \in W$ (resp. $uv^{-1} \in W$). In particular, for a given $w \in W$, the number of \uparrow_w -paths is equal to the number of reduced expressions of w .*

Some \uparrow -path and \downarrow -path on the permutohedron P_3 are given by Figure 17. Let us notice that the \uparrow -path at the left of Figure 17 is a $\uparrow_{s_1 s_2}$ -path and that it is the only one. In fact the only $w \in W$ such that the related \uparrow_w -path is not unique is $w = w_0$ and there are then two \uparrow_{w_0} -paths.

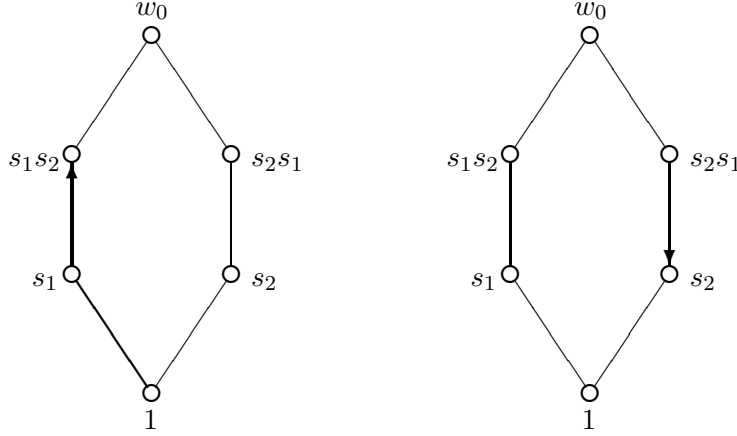


FIGURE 17. A \uparrow -path and a \downarrow -path on the W -permutohedron P_3

6.2. The set $R^\tau(w)$. It turns out, as seen in Remark 5.2, that we can define stable subsets $R^\tau(w)$ of double reduced words. In fact, their combinatorics, involving right τ -moves and generalized d -moves (or left τ -moves and generalized d -moves) is given by the W -permutohedron associated to \mathfrak{g} . In this subsection, we choose to focus on the right τ -moves, but the same combinatorics could be developed by considering left τ -moves.

Definition 6.2. Let \mathbf{i} be a double word. A *right d^τ -move* (or simply a *d^τ -move*) on \mathbf{i} is given by one of these transformations:

- a generalized d -move;
- a right τ -move.

For every $w \in W$, let $R^\tau(w)$ be the set of all the double words obtained from a word $\mathbf{i} \in R(1, w)$ by composition of d^τ -moves. (The choice of the double word \mathbf{i} doesn't matter, because of Theorem 3.1.)

It is easy to see that this definition coincides with the one given in Remark 5.2: the set $R^\tau(w)$ is the union of the disjoint sets $R(w'^{-1}, ww'^{-1})$ for every $w' \leq w \in W$, that is

$$(6.1) \quad R^\tau(w) = \bigcup_{w' \leq w \in W} R(w'^{-1}, ww'^{-1}).$$

(In particular, to every $\mathbf{i} \in R^\tau(w)$ there exists $w' \in W$ such that $\mathbf{i} \in R(w'^{-1}, ww'^{-1})$.)

In the same way Lemma 3.5 relates generalized d -moves to the set $R(u, v)$ of double reduced words associated to the elements $u, v \in W$, the link between d^τ -moves to the set $R^\tau(w)$ associated to $w \in W$ is the following, whose proof is immediate.

Lemma 6.3. A double reduced word $\mathbf{j} \in R^\tau(w)$ can be obtained from a double reduced word $\mathbf{i} \in R^\tau(w')$ by a sequence of d^τ -moves $\delta_{\mathbf{i} \rightarrow \mathbf{j}}^\tau$ if and only if the equality $w' = w$ on W is satisfied.

The link between double reduced words and the set $R^\tau(w)$ given by equation (6.1) can be strengthened by relating $R^\tau(w)$ to the vertices of the W -permutohedron in the following way. Let us recall that to $i \in [1, l] \cup [\bar{1}, \bar{l}]$, we can associate the positive letter $|i| \in [1, l]$ given by the formula (5.9). Let us denote, for every double word $\mathbf{i} = i_1 \dots i_n$, $\mathfrak{R}_{|i_n|}(\mathbf{i})$ (resp. $\mathfrak{L}_{|i_1|}(\mathbf{i})$) the result of a right (resp. left) τ -move on \mathbf{i} .

Lemma 6.4. *We have the following statements for every $w \in W$.*

- The set $R^\tau(w)$ is the disjoint union of the labels $R(w'^{-1}, ww'^{-1})$ associated to the vertices w' of the W -permutohedron P_W that are crossed or reached by a \uparrow_w -path.
- Two labeled vertices $R(u, v) \subset R^\tau(w)$ and $R(u', v') \subset R^\tau(w)$ of P_W are related by the edge s_j if and only if there exist double reduced words $\mathbf{i} \in R(u, v)$ and $\mathbf{j} \in R(u', v')$ such that $\mathbf{j} = \mathfrak{R}_j(\mathbf{i})$.

Proof. Let $W_w \subset W$ be the set of elements w' such that $w' \leq w$. Noticing that the map $W_w \rightarrow R^\tau(w) : w' \mapsto R(w'^{-1}, ww'^{-1})$ is a bijection for every $w \in W$, the first statement is just a translation of equation (6.1). So let us consider the second statement. It is easy to see that if double reduced words $\mathbf{i} \in R(u, v)$ and $\mathbf{j} \in R(u', v')$ are related by a right τ -move \mathfrak{R}_j , then there exist w_1 and w_2 such that $\mathbf{i} \in R(w_1^{-1}, w_0 w_1^{-1})$ and $\mathbf{j} \in R(w_2^{-1}, w_0 w_2^{-1})$ and $w_2 = w_1 s_j$ by using Lemma 5.27. Now, if the vertices w'_1 and w'_2 are linked by the edge s_j , we have the equality $w'_2 = w'_1 s_j$. The associated set of double reduced words are therefore $R(w'_1{}^{-1}, w_0 w'_1{}^{-1})$ and $R(s_j w'_1{}^{-1}, w_0 s_j w'_1{}^{-1})$, and we still use Lemma 5.27 to get some double reduced words \mathbf{i} and \mathbf{j} such that $\mathbf{j} = \mathfrak{R}_j(\mathbf{i})$. \square

Example 6.5. Let $\mathfrak{g} = A_2$. Figure 18 gives examples of sets $R^\tau(w)$ and their link with the permutohedron P_3 .

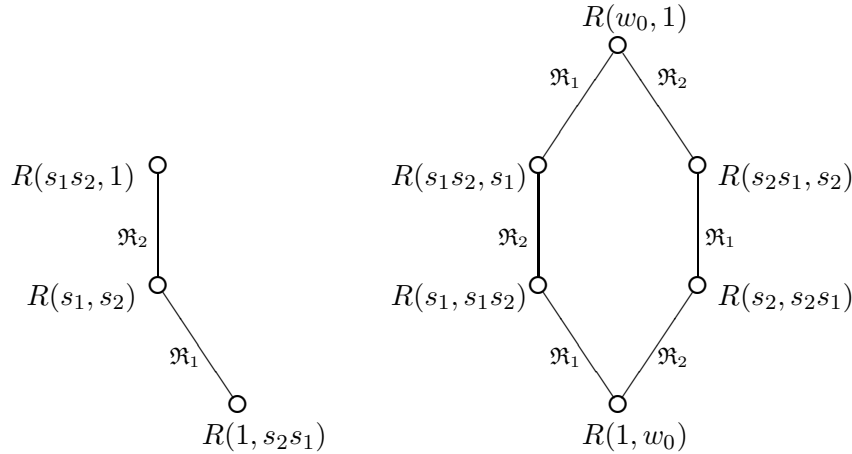


FIGURE 18. The sets $R^\tau(s_2s_1)$ and $R^\tau(w_0)$ when $\mathfrak{g} = A_2$

6.3. The family $\mathcal{X}^{\tau(w)}$ of cluster \mathcal{X} -varieties related to $G^{1,w}$. The same ideas can be applied at the level of cluster \mathcal{X} -varieties. Let us remember the notations of Definition 5.12. To any double reduced words $\mathbf{i}, \mathbf{i}' \in R^\tau(w)$ such that there exists a d^τ -move $\delta : \mathbf{i} \rightarrow \mathbf{i}'$ we associate the generalized cluster transformation $\mu_{\mathbf{i} \rightarrow \mathbf{i}'}^\tau : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{i}'}$ given by

- the cluster transformation $\mu_{\mathbf{i} \rightarrow \mathbf{i}'}$ if δ is a generalized d -move;

- the tropical mutation $\mu_{\diamond_i^{\mathfrak{R}}}$ if δ is the τ -move \mathfrak{R}_i .

We extend this definition to every $\mathbf{i}, \mathbf{j} \in R^\tau(w)$ in the following way. If \mathbf{i}, \mathbf{j} are double words linked by a sequence $\delta_{\mathbf{i} \rightarrow \mathbf{j}}^\tau$ of d^τ -moves and $\mathbf{i} \rightarrow \mathbf{i}_1 \rightarrow \cdots \rightarrow \mathbf{i}_{n-1} \rightarrow \mathbf{j}$ is the associated chain of elements, we define the map $\mu_{\mathbf{i} \rightarrow \mathbf{j}}^\tau$ as the composition $\mu_{\mathbf{i}_{n-1} \rightarrow \mathbf{j}}^\tau \circ \cdots \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}_1}^\tau$.

Because the birational Poisson isomorphism $\mu_{\mathbf{i} \rightarrow \mathbf{j}}^\tau : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{j}}$ associated to such a sequence $\delta_{\mathbf{i} \rightarrow \mathbf{j}}^\tau$ is a generalized cluster transformation for every $\mathbf{i}, \mathbf{j} \in R^\tau(w)$, we get a family of cluster \mathcal{X} -varieties $\mathcal{X}_{|\mathbf{i}|}$ associated to the set $R^\tau(w)$ and related by tropical mutations, that we denote \mathcal{X}_w^τ . The combinatorics is in fact encoded by the W -permutohedron P_W , as stated by the following result, straightforwardly deduced from Lemma 6.4.

Lemma 6.6. *Let $w \in W$. Replace each label $w' \in W$ of a vertex of the W -permutohedron P_W by the cluster \mathcal{X} -variety $\mathcal{X}^{w'^{-1}, ww'^{-1}}$. We then have the following properties.*

- The family \mathcal{X}_w^τ of cluster \mathcal{X} -varieties contains the cluster \mathcal{X} -variety $\mathcal{X}^{w'^{-1}, ww'^{-1}}$ associated to any $w' \in W$ of the W -permutohedron that can be crossed or reached by a \uparrow_w path.
- For every $i \in [1, l]$, if two vertices respectively related to the labels $\mathcal{X}^{u,v}, \mathcal{X}^{u',v'} \subset \mathcal{X}_w^\tau$ of P_W are related by the edge $s_i \in W$, then there exist two double reduced words $\mathbf{i}, \mathbf{j} \in R^\tau(w_0)$ such that the associated seed \mathcal{X} -tori $\mathcal{X}_{\mathbf{i}}$ and $\mathcal{X}_{\mathbf{j}}$ are related by the right tropical mutation $\mu_{\diamond_i^{\mathfrak{R}}}$ associated to i .

We finally define new evaluation maps to associate the family \mathcal{X}_w^τ to every double Bruhat cell $G^{w,1}$. To any $w \in W$ and any double word $\mathbf{i} \in R^\tau(w)$, we associate the evaluation map $\text{ev}_{\mathbf{i}}^\tau : \mathcal{X}_{\mathbf{i}} \rightarrow (G^{w,1}, \pi_G)$ by the formula

$$(6.2) \quad \text{ev}_{\mathbf{i}}^\tau(\mathbf{x}) = [\text{ev}_{\mathbf{i}}(\mathbf{x}) \widehat{w'w^{-1}}]_{\leq 0}, \quad \text{for every } \mathbf{x} \in \mathcal{X}_{\mathbf{i}}.$$

Lemma 6.7. *For every $w \in W$ and every $\mathbf{i} \in R^\tau(w)$ the map $\text{ev}_{\mathbf{i}}^\tau$ is a birational Poisson isomorphism on a Zarisky open set of $G^{w,1}$.*

Proof. Let us recall from [FZ99, Theorem 1.6] that the map $g \mapsto [g\widehat{w}]_{\leq 0}$ is biregular on $G^{1,w}$. Therefore the statement is implied by Theorem 3.4 and Proposition 5.11. \square

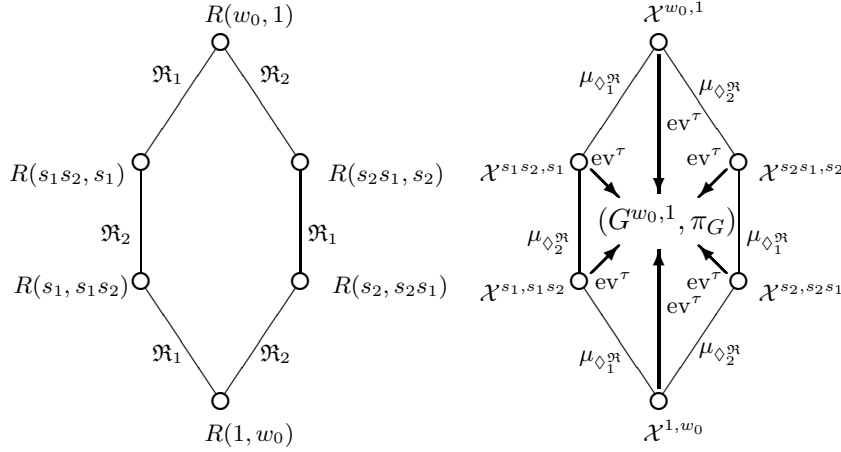
Lemma 6.8. *For every $w \in W$ and every $\mathbf{i}, \mathbf{j} \in R^\tau(w)$, the equality $\text{ev}_{\mathbf{i}}^\tau = \text{ev}_{\mathbf{j}}^\tau \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}}^\tau$ is satisfied.*

Proof. Derived from Theorem 3.7 and Proposition 5.24, involving respectively cluster transformations and tropical mutations. \square

When $\mathfrak{g} = A_2$, a synthesis of this section is provided by Figure 19.

7. TWISTED EVALUATIONS, $(w_1, w_2)_v$ -MAPS, AND CLUSTER VARIETIES RELATED TO (G, π_*)

We continue the evaluation procedure for dual Poisson-Lie groups started in Section 4. For every $v \in W$, we introduce a new set $D(v)$ of double words, containing the set $R(v, w_0)$. To each double word \mathbf{i} of this set, we associate a *twisted evaluation* $\widehat{\text{ev}}_{\mathbf{i}} : \mathcal{X}_{|\mathbf{i}|_{\mathfrak{R}}} \rightarrow G$, which then generalized the dual evaluation of Section 4. These twisted evaluations are obtained by composing the Fock-Goncharov evaluation maps of Section 3 with new maps called $(w_1, w_2)_v$ -maps, where for $w_1 = w_2 = e$, we rediscover the Evens-Lu morphisms [EL07, Section 5]. We then use the τ -combinatorics of Section 6 with the truncation maps of

FIGURE 19. The set $R^\tau(w_0)$ and related cluster \mathcal{X} -varieties when $\mathfrak{g} = A_2$

Section 4 to get a family of cluster \mathcal{X} -varieties \mathcal{X}_w , associated to each element w of W parameterizing (BB_-, π_*) . In particular, setting $w = e$, we rediscover the result of Section 4. (The way to relate these cluster \mathcal{X} -varieties will be given in Section 8 by introducing the birational Poisson isomorphisms on seed \mathcal{X} -tori called saltations.)

7.1. The $(w_1, w_2)_v$ -maps. The following $(w_1, w_2)_v$ -maps generalize the Poisson birational isomorphisms studied in [EL07, Section 5], which link direct products of double Bruhat cells and double reduced Bruhat cells to Steinberg fibers. Let us remember the involution $i \mapsto i^*$ on double word. Now, let $w \in W$ and $s_{i_1} \dots s_{i_n}$ be a reduced decomposition of w , then $w^* \in W$ is the element given by $w^* = s_{i_1^*} \dots s_{i_n^*}$. (Using the Tits theorem, it is easy to see that the result doesn't depend on the choice of the decomposition of w into simple reflections.) Remember the notation (4.1) for double reduced Bruhat cells. We denote $\pi_{G \times G}$ the Poisson product structure on the manifold $G \times G$ induced by the Poisson manifold (G, π_G) . For every $w_1 \leq v, w_2 \in W$, let $((G, L)^{(w_1, w_2)_v}, \pi_{(w_1, w_2)_v})$ be the quotient of the direct product $(G^{w_1^{*-1}, v w_1^{-1}} \times G^{w_2^{-1}, w_0 w_2^{-1}}, \pi_{G \times G})$ by the $H \times H$ -right action given by

$$(7.1) \quad \begin{aligned} & (g_1, g_2) \cdot (h_1, h_2) = (g_1 h_1, h_1^{-1} g_2 h_2) \\ & \text{with} \\ & g_1 \in G^{w_1^{*-1}, v w_1^{-1}}, \quad g_2 \in G^{w_2^{-1}, w_0 w_2^{-1}}, \quad \text{and } h_1, h_2 \in H. \end{aligned}$$

In particular, when $w_1 = v$ and $w_2 = e$, the quotient set $(G, L)^{(w_1, w_2)_v}$ is the set L^{v, w_0} .

Definition 7.1. Let $w_1 \leq v, w_2 \in W$. The $(w_1, w_2)_v$ -right map $\varrho_{(w_1, w_2)_v}$ and the $(w_1, w_2)_v$ -left map $\lambda_{(w_1, w_2)_v}$ are defined by the following formulas

$$\begin{aligned} \varrho_{(w_1, w_2)_v} : (G, L)^{(w_1, w_2)_v} &\rightarrow L^{w_0, v w_1^{-1}} : (b_1, bH) \mapsto b_1 [b \widehat{w_2 w_0}]_{\leq 0} H, \\ \text{and} \\ \lambda_{(w_1, w_2)_v} : (G, L)^{(w_1, w_2)_v} &\rightarrow L^{w_1^{*-1}, w_0} : (b_1, bH) \mapsto b_1 [[b \widehat{w_2 w_0}]_{\leq 0}^{\theta} \widehat{w_0}]_{\leq 0}^{\theta} H, \end{aligned}$$

where θ denotes the Cartan involution on G given by (5.8). For every $t \in H$, the $(w_1, w_2)_v$ -maps (or simply (w_1, w_2) -maps when no confusion occurs) are then the maps given by

$$\rho_{t, (w_1, w_2)_v} : (G, L)^{(w_1, w_2)_v} \rightarrow G : (b_1, bH) \mapsto \lambda_{(w_1, w_2)_v}(b_1, bH) \widehat{w_0} t \varrho_{(w_1, w_2)_v}(b_1, bH)^{-1}.$$

Example 7.2. The $(w_1, w_2)_v$ -maps associated to $\mathfrak{g} = A_2$ are obtained in the following way. Let us first describe, for every $v \in W$, the set $W_{\leq v}$ of elements $w_1 \in W$ such that $w_1 \leq v$. They are the following: $W_{\leq e} = \{e\}$, $W_{\leq s_i} = \{e, s_i\}$, $W_{\leq s_i s_j} = \{e, s_i, s_i s_j\}$ and $W_{\leq s_i s_j s_i} = W$, $i, j \in [1, 2]$ being different numbers. Then, let $g_1 \in G^{w_1^{-1}, v w_1^{-1}}$, $b \in G^{1, w_0}$, $g \in G^{s_i, s_i s_j}$, $g' \in G^{s_i s_j, s_i}$ and $c \in G^{w_0, 1}$; the different $(w_1, w_2)_v$ -maps are given by

$$\begin{aligned} \rho_{t, (w_1, e)_v}(g_1, b) &= g_1 b t \widehat{w_0} (g_1 [b \widehat{w_0}]_{\leq 0})^{-1}; \\ \rho_{t, (w_1, s_i)_v}(g_1, g) &= g_1 [[g \widehat{s_i}]_{\leq 0}^{\theta} \widehat{w_0}]_{\leq 0}^{\theta} t \widehat{w_0} (g_1 [g \widehat{s_i}]_{\leq 0})^{-1}; \\ \rho_{t, (w_1, s_i s_j)_v}(g_1, g') &= g_1 [[g' \widehat{s_i}]_{\leq 0}^{\theta} \widehat{w_0}]_{\leq 0}^{\theta} t \widehat{w_0} (g_1 [g' \widehat{s_i}]_{\leq 0})^{-1}; \\ \rho_{t, (w_1, w_0)_v}(g_1, c) &= g_1 [c^{\theta} \widehat{w_0}]_{\leq 0}^{\theta} t \widehat{w_0} (g_1 c)^{-1}. \end{aligned}$$

The way to relate the geometries of (G, π_G) and (G, π_*) is given by the following result, which is directly deduced from Theorem 3.4, Lemma 7.8 and the forthcoming Theorem 7.9.

Proposition 7.3. *For every $t \in H$ and $w_1 \leq v, w_2 \in W$, the $(w_1, w_2)_v$ -map $\rho_{t, (w_1, w_2)_v}$ is a birational Poisson isomorphism of $((G, L)^{(w_1, w_2)_v}, \pi_{(w_1, w_2)_v})$ on a Zariski open set of $(F_{t, w_0 v^{-1}}, \pi_*)$.*

Remark 7.4. For every $v \in W$, the $(e, e)_v$ -maps are the maps denoted $\mu\rho$ in [EL07, Section 5] and Proposition 7.3 below then rephrases [EL07, Corollary 5.11]. Another interesting case is given by $w_1 = v$ and $w_2 = e$; in this case, the related $(w_1, w_2)_v$ -maps are the following maps, strongly related to the dual evaluations of Section 4.

$$\begin{aligned} \rho_{t, (v, e)_v} : (L^{v, w_0}, \pi_{G \setminus H}) &\rightarrow (F_{t, w_0 v^{-1}}, \pi_*) \\ gH &\mapsto g \widehat{w_0} t [g \widehat{w_0}]_{\leq 0}^{-1}. \end{aligned}$$

Indeed, the equality $\text{ev}_{\mathbf{i}}^{\text{dual}} = \rho_{t, (v, e)_v} \circ \text{ev}_{\mathbf{i}}^{\text{red}}$ is clearly satisfied on $\mathcal{X}_{[\mathbf{i}]\mathfrak{R}}(t)$ for every $t \in H$ and every $\mathbf{i} \in R(v, w_0)$.

7.2. Twisted evaluations on (G, π_*) . As it will be stated by Lemma 7.8, a composition of evaluations and reduced evaluations with $(w_1, w_2)_v$ -maps leads to a generalization of dual evaluations, called twisted evaluations and given by the formula (7.4). But, before, we introduce new sets of double words, denoted $W(w_1, w_2)_v$, and a new operation on seed \mathcal{X} -tori called \mathcal{X} -split, as preliminaries to the definition of twisted evaluations.

Definition 7.5. Let $w_1 \leq v, w_2 \in W$. A $(w_1, w_2)_v$ -word \mathbf{i} is a double word linked to a product $\mathbf{i}_1 \mathbf{i}_2$, with $\mathbf{i}_1 \in R(w_1^{-1}, v w_1^{-1})$ and $\mathbf{i}_2 \in R(w_2^{-1}, w_0 w_2^{-1})$, by a sequence of mixed 2-moves. The product $\mathbf{i}_1 \mathbf{i}_2$ is called a *trivial $(w_1, w_2)_v$ -word* and the decomposition $(\mathbf{i}_1, \mathbf{i}_2)$ associated to \mathbf{i} is called the *$(w_1, w_2)_v$ -decomposition* of \mathbf{i} . (For example, every $(e, e)_v$ -word is a trivial $(e, e)_v$ -word.) The set of $(w_1, w_2)_v$ -words will be denoted $W(w_1, w_2)_v$.

In particular, the set $R(v, w_0)$ is the set of $(v, e)_v$ -words. Let $D(v)$ be the (disjoint) union over $w_1 \leq v, w_2 \in W$ of all the $(w_1, w_2)_v$ -words.

$$D(v) = \bigsqcup_{w_1 \leq v, w_2 \in W} W(w_1, w_2)_v.$$

Therefore, we have the inclusion $R(v, w_0) \subset D(v)$ for every $v \in W$.

A complete description of the sets $D(v)$ will be given in Subsection 8.1 via the W -permutohedron. Here is, for the moment, an example with $\mathfrak{g} = A_2$ and $v = s_1$. The associated sets $R(w_1^{\star-1}, vw_1^{-1})$ and $R(w_2^{-1}, w_0w_2^{-1})$ are then given by the following list.

$$R(1, s_1), R(s_2, 1) \quad \text{and} \quad R(1, w_0), R(s_1, s_1s_2), R(s_1s_2, s_1), \\ R(s_2, s_2s_1), R(s_2s_1, s_2), R(w_0, 1) .$$

Trivial $(w_1, w_2)_v$ -words are therefore products $\mathbf{i}_1\mathbf{i}_2$ of some double words

$$\mathbf{i}_1 \in \{1, \bar{2}\} \quad \text{and} \quad \mathbf{i}_2 \in \{121, 212, \bar{1}1\bar{2}, 1\bar{1}2, 12\bar{1}, \bar{1}21, \bar{1}1\bar{2}, 1\bar{1}2, \bar{2}21, \bar{2}21, 21\bar{2}, \bar{2}1\bar{2}, \\ \bar{2}2\bar{1}, \bar{2}2\bar{1}, \bar{1}2\bar{1}, \bar{2}1\bar{2}\} .$$

We then introduce a new operation on seed \mathcal{X} -tori that will be useful to describe the oncoming combinatorics related to the following twisted evaluations.

Definition 7.6. A *split* of a seed \mathbf{I} is a pair of seeds $(\mathbf{I}_1, \mathbf{I}_2)$ such that \mathbf{I} is their amalgamated product, that is $\mathbf{I} = \mathbf{m}(\mathbf{I}_1, \mathbf{I}_2)$. An associated \mathcal{X} -*split* is a section of the amalgamation map $\mathbf{m} : \mathcal{X}_{\mathbf{I}_1} \times \mathcal{X}_{\mathbf{I}_2} \rightarrow \mathcal{X}_{\mathbf{I}}$, i.e. a map $\mathfrak{s} : \mathcal{X}_{\mathbf{I}} \rightarrow \mathcal{X}_{\mathbf{I}_1} \times \mathcal{X}_{\mathbf{I}_2}$ such that the product $\mathbf{m} \circ \mathfrak{s}$ gives the identity map on $\mathcal{X}_{\mathbf{I}}$. For every \mathcal{X} -split \mathfrak{s} associated to the decomposition $\mathbf{I} \rightarrow (\mathbf{I}_1, \mathbf{I}_2)$, we will associate to any $\mathbf{x} \in \mathcal{X}_{\mathbf{I}}$, the elements $\mathbf{x}_{(1)} \in \mathcal{X}_{\mathbf{I}_1}$ and $\mathbf{x}_{(2)} \in \mathcal{X}_{\mathbf{I}_2}$ given by

$$\mathfrak{s}(\mathbf{x}) = (\mathbf{x}_{(1)}, \mathbf{x}_{(2)}) .$$

Figure 20 describes the split associated to the decomposition $2\bar{2} \rightarrow (2, \bar{2})$ when $\mathfrak{g} = A_3$. Let us stress that, because the amalgamated product is not an isomorphism of seed \mathcal{X} -tori,

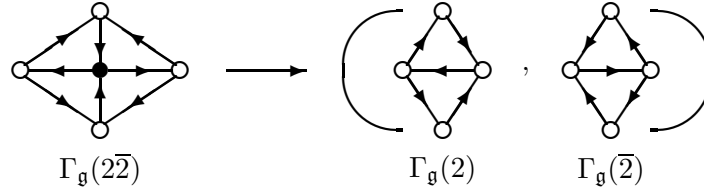


FIGURE 20. The split $\mathfrak{s} : \Gamma_{\mathfrak{g}}(2\bar{2}) \mapsto (\Gamma_{\mathfrak{g}}(2), \Gamma_{\mathfrak{g}}(\bar{2}))$ for $\mathfrak{g} = A_3$

different \mathcal{X} -splits can be associated to a given split of seed. Indeed, let $\mathbf{i}, \mathbf{i}_1, \mathbf{i}_2$ be double words such that $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$, if \mathfrak{s} is a \mathcal{X} -split associated to the decomposition $\mathbf{i} \rightarrow (\mathbf{i}_1, \mathbf{i}_2)$, then for every $\mathbf{t} \in \mathcal{X}_{\mathbf{I}}$ the following map $\mathfrak{s}_{\mathbf{t}}$ is also a \mathcal{X} -split.

$$\mathfrak{s}_{\mathbf{t}}(\mathbf{x}) = (\mathbf{m}(\mathbf{x}_{(1)}, \mathbf{t}), \mathbf{m}(\mathbf{t}^{-1}, \mathbf{x}_{(2)})) \quad \text{where} \quad \mathfrak{s}(\mathbf{x}) = (\mathbf{x}_{(1)}, \mathbf{x}_{(2)})$$

$$\text{and } \mathbf{t}^{-1} = (t_1^{-1}, \dots, t_l^{-1}) \quad \text{when} \quad \mathbf{t} = (t_1, \dots, t_l) .$$

However, in what follows, nearly each time we will need a \mathcal{X} -split, this freedom of choice will not affect the related result. We are now ready to define twisted evaluations on (G, π_*) .

Definition 7.7. Let $w_1 \leq v, w_2 \in W$, $\mathbf{i} = \mathbf{i}_1\mathbf{i}_2$ be a trivial $(w_1, w_2)_v$ -word and \mathfrak{s} be a \mathcal{X} -split associated to the decomposition $\mathbf{i} \rightarrow (\mathbf{i}_1, \mathbf{i}_2)$. We define the *left* and *right evaluations* $\text{ev}_{\mathbf{i}}^{\mathcal{L}}, \text{ev}_{\mathbf{i}}^{\mathcal{R}} : \mathcal{X}_{\mathbf{i}} \rightarrow G$ by the formulas:

$$(7.2) \quad \begin{aligned} \text{ev}_{\mathbf{i}}^{\mathcal{R}}(\mathbf{x}) &= \text{ev}_{\mathbf{i}_1}(\mathbf{x}_{(1)})[\text{ev}_{\mathbf{i}_2}^{\text{red}}(\mathbf{x}_{(2)})\widehat{w_2w_0}]_{\leq 0} \\ \text{and} \quad \text{ev}_{\mathbf{i}}^{\mathcal{L}}(\mathbf{x}) &= \text{ev}_{\mathbf{i}_1}(\mathbf{x}_{(1)})[(\text{ev}_{\mathbf{i}_2}^{\mathcal{R}}(\mathbf{x}_{(2)}))^{\theta}\widehat{w_0}]_{\leq 0}^{\theta} , \end{aligned}$$

where the reduced evaluations ev^{red} are the evaluation maps given in Section 4. (Let us notice that these left and right evaluations don't depend on the choice of the \mathcal{X} -split \mathfrak{s} .) These maps are extended to every $\mathbf{i} \in D(v)$ by setting

$$(7.3) \quad \text{ev}_{\mathbf{i}}^{\mathfrak{L}} = \text{ev}_{\mathbf{i}_1 \mathbf{i}_2}^{\mathfrak{L}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}_1 \mathbf{i}_2} \quad \text{and} \quad \text{ev}_{\mathbf{i}}^{\mathfrak{R}} = \text{ev}_{\mathbf{i}_1 \mathbf{i}_2}^{\mathfrak{R}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}_1 \mathbf{i}_2} .$$

Finally, for every $v \in W$ and $\mathbf{i} \in D(v)$, we define the *twisted evaluation*

$$(7.4) \quad \widehat{\text{ev}}_{\mathbf{i}} : \mathcal{X}_{[\mathbf{i}]} \rightarrow G : \mathbf{x} \mapsto \text{ev}_{\mathbf{i}}^{\mathfrak{L}}(\mathbf{x}) x_1(\mathbf{x}(\mathfrak{R})) \widehat{w}_0 \text{ev}_{\mathbf{i}}^{\mathfrak{R}}(\mathbf{x})^{-1} ,$$

where $\text{ev}_1(\mathbf{x}(\mathfrak{R})) = \prod_{j=1}^l H^j(x_{(N^j(\mathbf{i}))}) .$

The relations between twisted evaluations, (w_1, w_2) -maps and the Fock-Goncharov evaluation maps is given by the following lemma, straightforwardly checked.

Lemma 7.8. *Let $w_1 \leq v, w_2 \in W$, \mathbf{i} be a trivial $(w_1, w_2)_v$ -word, and \mathfrak{s} be a \mathcal{X} -split associated to the $(w_1, w_2)_v$ -decomposition $\mathbf{i} \rightarrow (\mathbf{i}_1, \mathbf{i}_2)$. The following equalities are satisfied.*

$$\text{ev}_{\mathbf{i}}^{\mathfrak{R}}(\mathbf{x}) = \varrho_{(w_1, w_2)_v}(\text{ev}_{\mathbf{i}_1}(\mathbf{x}_{(1)}), \text{ev}_{\mathbf{i}_2}^{\text{red}}(\mathbf{x}_{(2)})) ,$$

$$\text{ev}_{\mathbf{i}}^{\mathfrak{L}}(\mathbf{x}) = \lambda_{(w_1, w_2)_v}(\text{ev}_{\mathbf{i}_1}(\mathbf{x}_{(1)}), \text{ev}_{\mathbf{i}_2}^{\text{red}}(\mathbf{x}_{(2)})) ,$$

$$\widehat{\text{ev}}_{\mathbf{i}}(\mathbf{x}) = \rho_{\text{ev}_1(\mathbf{x}(\mathfrak{R})), (w_1, w_2)_v}(\text{ev}_{\mathbf{i}_1}(\mathbf{x}_{(1)}), \text{ev}_{\mathbf{i}_2}^{\text{red}}(\mathbf{x}_{(2)})) .$$

Here is finally the analog of Theorem 3.4 for (G, π_*) .

Theorem 7.9. *For every $v \in W$, $t \in H$ and $\mathbf{i} \in D(v)$, the restriction $\widehat{\text{ev}}_{\mathbf{i}} : \mathcal{X}_{[\mathbf{i}]}(t) \rightarrow (F_{t, w_0 v^{-1}}, \pi_*)$ is a Poisson birational isomorphism onto a Zariski open set $F_{t, w_0 v^{-1}}^0$ of $F_{t, w_0 v^{-1}}$.*

Proof. We are going to use the three following lemmas, the first one being crucial to our construction.

Lemma 7.10. [EL07, Corollary 5.11] *Let $v \in W$. The $(e, e)_v$ -map associated to every $t \in H$ is a birational Poisson isomorphism of $((G, L)^{(e, e)_v}, \pi_{(e, e)_v})$ on a Zariski open set of $(F_{t, w_0 v^{-1}}, \pi_*)$.*

Lemma 7.11. *Let $t \in H$, $v \in W$, \mathbf{i} be a $(e, e)_v$ -word and \mathfrak{s} be a \mathcal{X} -split associated to the $(e, e)_v$ -decomposition $\mathbf{i} \rightarrow (\mathbf{i}_1, \mathbf{i}_2)$. The following evaluation map is a birational Poisson isomorphism onto a Zariski open set $F_{t, w_0 v^{-1}}^0 \subset F_{t, w_0 v^{-1}}$.*

$$\widehat{\text{ev}}_{t, \mathbf{i}} : \mathcal{X}_{\mathbf{i}}^{\text{red}} \rightarrow (F_{t, w_0 v^{-1}}, \pi_*) : \mathbf{x} \mapsto \text{ev}_{\mathbf{i}}^{\text{red}}(\mathbf{x}) \widehat{w}_0 t (\text{ev}_{\mathbf{i}_1}(\mathbf{x}_{(1)}) [\text{ev}_{\mathbf{i}_2}^{\text{red}}(\mathbf{x}_{(2)}) \widehat{w}_0]_{\leq 0})^{-1} .$$

Lemma 7.11 is easily deduced from Lemma 7.10 and Theorem 3.7.

Lemma 7.12. *The map $b \mapsto [b \widehat{w}_0]_{\leq 0}^{\theta}$ is an involution on G^{e, w_0} .*

Lemma 7.12 is well-known; to prove it, we have to remember that the map θ is an involution, and use successively the facts that $\widehat{w}_0^{\theta} = \widehat{w}_0^{-1}$, $b^{\theta} \in G^{w_0, e}$, and $\widehat{w}_0^{-1} [b \widehat{w}_0]_{+}^{\theta} \widehat{w}_0 \in N$ to get

$$b^{\theta} = (b \widehat{w}_0)^{\theta} \widehat{w}_0 = [(b \widehat{w}_0)^{\theta} \widehat{w}_0]_{\leq 0} = [[b \widehat{w}_0]_{\leq 0}^{\theta} \widehat{w}_0]_{\leq 0} .$$

Let us now choose $\mathbf{x}(\mathfrak{R}) \in \mathcal{X}_1$ such that $\text{ev}_1(\mathbf{x}(\mathfrak{R})) = t$. From Lemma 7.11 and Lemma 7.12, we deduce that when the $(w_1, w_2)_v$ -word $\mathbf{i} \in D(v)$ is such that $w_1 = w_2 = e$ the

evaluation $\widehat{\text{ev}}_{t,\mathbf{i}}$ gives the restriction of the map $\widehat{\text{ev}}_{\mathbf{i}}$ on the set $\mathcal{X}_{[\mathbf{i}]\mathfrak{R}}(t)$. The general case will be deduced from Theorem 8.12 by noticing that the associated map $\widehat{\mu}_{\mathbf{i} \rightarrow \mathbf{j}}$, defined in Section 8, is a birational Poisson isomorphism for every double words $\mathbf{i}, \mathbf{j} \in D(v)$. \square

In particular, Theorem 4.10 is deduced from Theorem 7.9 because $\widehat{\text{ev}}_{\mathbf{i}}$ and $\text{ev}_{\mathbf{i}}^{\text{dual}}$ coincide for every $\mathbf{i} \in R(v, w_0)$, that is for every $(w_1, w_2)_v$ -word \mathbf{i} such that $w_1 = v$ and $w_2 = e$. Finally, let us recall that for every double word \mathbf{i} , $\mathcal{X}_{\mathbf{i}}^{\text{dual}} \subset \mathcal{X}_{[\mathbf{i}]\mathfrak{R}}$ is, as a set, such that the elements of the set of variables $\mathbf{x}(\mathfrak{R})$ are pairwise disjoint. Thus, we get the following corollary from the second decomposition of (2.11) with Theorem 7.9.

Corollary 7.13. *For every $\mathbf{i} \in D(w_0)$, the map $\widehat{\text{ev}}_{\mathbf{i}} : \mathcal{X}_{\mathbf{i}}^{\text{dual}} \rightarrow (BB_-, \pi_*)$ is a Poisson birational isomorphism on a Zariski open set of BB_- .*

Remark 7.14. The same construction can be done (with the same results) if G is no more of adjoint type but simply connected. In this case, the twisted evaluations we have to consider are the following

$$\begin{aligned} \widehat{\text{ev}}_{\mathbf{i}} : \mathcal{X}_{[\mathbf{i}]\mathfrak{R}} &\rightarrow G : \mathbf{x} \mapsto \text{ev}_{\mathbf{i}}^{\mathfrak{S}}(\mathbf{x}) x_1(\mathbf{x}(\mathfrak{R})) \widehat{w_0} \text{ev}_{\mathbf{i}}^{\mathfrak{R}}(\mathbf{x})^{-1}, \\ \text{where } x_1(\mathbf{x}(\mathfrak{R})) &= \prod_{j=1}^l H_j(x_{\binom{j}{N^j(\mathbf{i})}}), \end{aligned}$$

the generators $H_j(\cdot)$ being given by (2.3).

7.3. τ -combinatorics and cluster \mathcal{X} -varieties related to (G, π_*) . We now relate twisted evaluations by cluster transformations to get Poisson parameterizations related to (G, π_*) by cluster \mathcal{X} -varieties. This is achieved by mixing the truncation maps of Section 4 with the τ -combinatorics developed in Section 6. We then get a family of \mathcal{X} -varieties \mathcal{X}_w , indexed by the Weyl group W of G , evaluating the dual Poisson Lie-group (BB_-, π_*) .

7.3.1. Double reduced words, the set $D_{w_1}(v)$, and the W -permutohedron associated to \mathfrak{g} . As it was done in Section 3, we start by the study of the combinatorics on double words before to consider the related birational Poisson isomorphisms on seed \mathcal{X} -tori. We fix $w_1 \leq v \in W$, and denote $D_{w_1}(v)$ the set of $(w_1, w_2)_v$ -words for every $w_2 \in W$. Therefore, the set $D(v)$ is the union over all $w_1 \leq v \in W$ of the sets $D_{w_1}(v)$.

$$(7.5) \quad D_{w_1}(v) = \bigcup_{w_2 \in W} W(w_1, w_2)_v \quad \text{and} \quad D(v) = \bigcup_{w_1 \leq v} D_{w_1}(v).$$

The following result is clear from Definition 7.5. It uses amalgamation to relate these sets $D_{w_1}(v)$ to the set $R^\tau(w_0)$ already studied in the Section 6.

Lemma 7.15. *The set $D_{w_1}(v)$ is the set of double words that can be obtained by a composition of mixed 2-moves from an amalgamation of a double reduced word $\mathbf{i}_1 \in R(w_1^{\star-1}, vw_1^{-1})$ with any double reduced word $\mathbf{i}_2 \in R^\tau(w_0)$.*

We can therefore use the fact that the set $R^\tau(w)$ has been described in Section 6 via the W -permutohedron to relate the sets $W(w_1, w_2)_v$ to $D_{w_1}(v)$, for every $w_1 \leq v \in W$.

Lemma 7.16. *We have the following statements for every $v, w_1 \in W$ such that $w_1 \leq v$.*

- *The set $D_{w_1}(v)$ is the disjoint union of the labels $W(w_1, w'^{-1})_v$ associated to the vertices w' of the W -permutohedron P_W that are crossed or reached by a \uparrow_v -path.*

- Two labeled vertices $W(w_1, w^{-1})_v, W(w_1, w'^{-1})_v \subset D_{w_1}(v)$ of P_W are related by the edge s_j if and only if there exist double reduced words $\mathbf{i} \in W(w_1, w^{-1})_v$ and $\mathbf{j} \in W(w_1, w'^{-1})_v$ such that $\mathbf{j} = \mathfrak{R}_j(\mathbf{i})$.

Proof. It suffices to apply Lemma 6.4 and the fact that the map $R(w'^{-1}, vw'^{-1}) \mapsto W(w_1, w'^{-1})_v$ gives a bijection from $R^\tau(v)$ to $D_{w_1}(v)$. \square

Example 7.17. We still take $\mathfrak{g} = A_2$. From Figure 18, we get the W -permutohedron at the right of Figure 21. Let us stress that this picture is valid for *any* $w_1 \in W$.

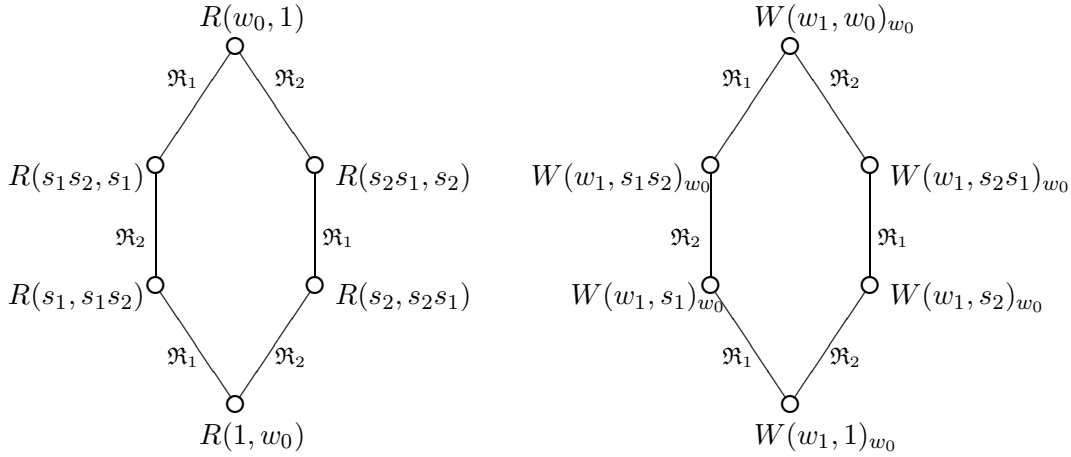


FIGURE 21. From the set $R^\tau(w_0)$ to the set $D_{w_1}(w_0)$ when $\mathfrak{g} = A_2$

7.3.2. *The cluster \mathcal{X} -varieties $\mathcal{X}_{w_1 \leq v}(t)$ associated to the set $D_{w_1}(v)$.* We now consider the related cluster transformations and associate a right truncated cluster \mathcal{X} -variety $\mathcal{X}_{w_1 \leq v}$ to each set $D_{w_1}(v)$. Let $w_1 \leq v \in W$. To any d^τ -move δ linking two double words $\mathbf{i}, \mathbf{i}' \in D_{w_1}(v)$ we associate the cluster transformation $\mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{i}']_{\mathfrak{R}}} : \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}} \rightarrow \mathcal{X}_{[\mathbf{i}']_{\mathfrak{R}}}$ such that

$$\mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{i}']_{\mathfrak{R}}} \circ \mathbf{t}_{\mathfrak{R}}(t) = \mathbf{t}_{\mathbf{i}'}^\tau(t) \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}'}^\tau,$$

where, as in Subsection 6.3, the generalized cluster $\mu_{\mathbf{i} \rightarrow \mathbf{j}}^\tau$ denotes

- the cluster transformation $\mu_{\mathbf{i} \rightarrow \mathbf{i}'}$ if δ is a generalized d -move;
- the tropical mutation $\mu_{\diamond_j^{\mathfrak{R}}}^{(i_{\ell(w)})}$ if δ is the τ -move \mathfrak{R}_j .

As usual, we extend this definition to every $\mathbf{i}, \mathbf{j} \in D_{w_1}(v)$: if \mathbf{i}, \mathbf{j} are double words linked by a sequence $\delta_{\mathbf{i} \rightarrow \mathbf{j}}^\tau$ of d^τ -moves and $\mathbf{i} \rightarrow \mathbf{i}_1 \rightarrow \cdots \rightarrow \mathbf{i}_{n-1} \rightarrow \mathbf{j}$ is the associated chain of elements, we define the map $\mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{j}]_{\mathfrak{R}}} : \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}} \rightarrow \mathcal{X}_{[\mathbf{j}]_{\mathfrak{R}}}$ as the composition

$$\mu_{[\mathbf{i}_{n-1}]_{\mathfrak{R}} \rightarrow [\mathbf{j}]_{\mathfrak{R}}} \circ \cdots \circ \mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{i}_1]_{\mathfrak{R}}}.$$

We will denote $\mathcal{X}_{w_1 \leq v}$, or simply \mathcal{X}_w if the equality $v = w_0$ is satisfied, the cluster \mathcal{X} -variety associated to the set $D_{w_1}(v)$. Moreover, because of equation (4.5), the cluster \mathcal{X} -variety $\mathcal{X}_{w_1 \leq v}$ can be Poisson stratified into the disjoint union over H of cluster \mathcal{X} -varieties $\mathcal{X}_{w_1 \leq v}(t)$:

$$\mathcal{X}_{w_1 \leq v} = \bigsqcup_{t \in H} \mathcal{X}_{w_1 \leq v}(t).$$

$$\mathfrak{m} : \mathcal{X}^{w_1^{-1}, vw_1^{-1}} \times \mathcal{X}^{w_2^{-1}, vw_2^{-1}} \rightarrow \mathcal{X}_{(w_1, w_2)_v} \ .$$

Lemma 7.19. *Replace each label $w' \in W$ of a vertex of the W -permutohedron P_W by the cluster \mathcal{X} -variety $\mathcal{X}^{w'^{-1}, w_0 w'^{-1}}$. We then have the following properties.*

- The cluster \mathcal{X} -variety \mathcal{X}_w , indexed by any element $w \in W$, contains the seed \mathcal{X} -torus $\mathcal{X}_{[\mathbf{i}]}_{\mathfrak{R}}$ associated to any double word $\mathbf{i} \in D_w(w_0)$.
- The cluster \mathcal{X} -variety $\mathcal{X}_{w_1 \leq v}$ is the image of the cluster \mathcal{X} -variety $\mathcal{X}_{(w_1, w')_v}$ by the right truncation map $\mathbf{t}_{\mathfrak{R}}$ for every $w' \in W$.
- For every $i \in [1, l]$, if two vertices respectively labeled by the cluster \mathcal{X} -varieties $\mathcal{X}^{u,v}, \mathcal{X}^{u',v'} \subset \mathcal{X}_{w_0}$ of P_W are related by the edge $s_i \in W$, then there exist two double reduced words $\mathbf{i}, \mathbf{j} \in D_{w_1}(v)$ such that the associated seed \mathcal{X} -tori $\mathcal{X}_{\mathbf{i}}$ and $\mathcal{X}_{\mathbf{j}}$ are related by the right tropical mutation associated to i and denoted $\mu_{\diamond_i^{\mathfrak{R}}}$.

This result is illustrated by Figure 22 in the case $\mathfrak{g} = A_2$.

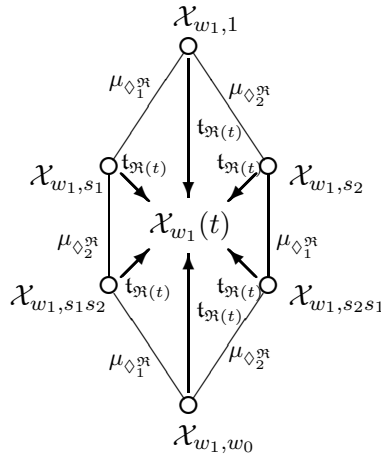


FIGURE 22. Truncation maps and the cluster \mathcal{X} -variety $\mathcal{X}_{w_1}(t)$ when $\mathfrak{g} = A_2$

7.3.3. Twisted evaluation maps relating $\mathcal{X}_{w_1 \leq v}(t)$ to $(F_{t,v}, \pi_*)$. We finally use twisted evaluation maps to get, for every $v \in W$, a family of truncated cluster \mathcal{X} -varieties $\mathcal{X}_{w_1 \leq v}(t)$, $w_1 \leq v \in W$, parameterizing the Poisson submanifold $(F_{t,w_0 v^{-1}}, \pi_*)$. In particular, the family of truncated cluster \mathcal{X} -varieties $\mathcal{X}_{w_1 \leq w_0}$ (also denoted \mathcal{X}_{w_1}), $w_1 \in W$, will parameterize the dual Poisson-Lie group (BB_-, π_*) .

Proposition 7.20. *For every $t \in H$ and $w_1 \leq v \in W$ and every double words $\mathbf{i}, \mathbf{i}' \in D_{w_1}(v)$, the equality $\widehat{\text{ev}}_{\mathbf{i}} = \widehat{\text{ev}}_{\mathbf{i}'} \circ \mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{i}']_{\mathfrak{R}}}$ is satisfied.*

Proof. We introduce relatives of the left and right evaluations of (7.2) in the following way. Let $w_1 \leq v, w_2 \in W$ and \mathfrak{s} be a \mathcal{X} -split associated to the $(w_1, w_2)_v$ -decomposition $\mathbf{i} \mapsto (\mathbf{i}_1, \mathbf{i}_2)$. We define the maps $\overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{L}}, \overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{R}} : \mathcal{X}_{\mathbf{i}} \rightarrow G$ and $\overline{\text{ev}}_{\mathbf{i}} : \mathcal{X}_{\mathbf{i}} \times \mathcal{X}_{\mathbf{1}} \rightarrow G$ by the following formulas: we start by a definition on trivial double words

$$\overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{R}}(\mathbf{x}) = \text{ev}_{\mathbf{i}_1}(\mathbf{x}_{(1)})[\text{ev}_{\mathbf{i}_2}(\mathbf{x}_{(2)})\widehat{w_2 w_0}]_{\leq 0}, \text{ and } \overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{L}}(\mathbf{x}) = \text{ev}_{\mathbf{i}_1}(\mathbf{x}_{(1)})[(\overline{\text{ev}}_{\mathbf{i}_2}^{\mathfrak{R}}(\mathbf{x}_{(2)}))^{\theta}\widehat{w_0}]_{\leq 0}^{\theta},$$

before to use the same idea as in (7.3) to extend this definition for every $\mathbf{i} \in D_{w_1}(v)$ by setting $\overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{L}} = \overline{\text{ev}}_{\mathbf{i}_1 \mathbf{i}_2}^{\mathfrak{L}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}_1 \mathbf{i}_2}$ and $\overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{R}} = \overline{\text{ev}}_{\mathbf{i}_1 \mathbf{i}_2}^{\mathfrak{R}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}_1 \mathbf{i}_2}$, and we introduce finally

$$\overline{\text{ev}}_{\mathbf{i}}(\mathbf{x}, \mathbf{t}) \mapsto \overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{L}}(\mathbf{x}) \widehat{w_1} \overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{R}}(\mathbf{x})^{-1}.$$

Lemma 6.8 with the extension of the kind (7.3) just described then implies that the following equalities are satisfied for every double words $\mathbf{i}, \mathbf{i}' \in D_{w_1}(v)$.

$$\overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{R}} = \overline{\text{ev}}_{\mathbf{i}'}^{\mathfrak{R}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}'}^{\tau} \quad , \quad \overline{\text{ev}}_{\mathbf{i}}^{\mathfrak{L}} = \overline{\text{ev}}_{\mathbf{i}'}^{\mathfrak{L}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}'}^{\tau} \quad \text{and} \quad \overline{\text{ev}}_{\mathbf{i}} = \overline{\text{ev}}_{\mathbf{i}'}(\mu_{\mathbf{i} \rightarrow \mathbf{i}'}^{\tau}(\mathbf{x}), \mathbf{t}).$$

Moreover, it is easy to see that the element $\overline{\text{ev}}_{\mathbf{i}}(\mathbf{x}, \mathbf{t})$ doesn't depend on x_j when $j \in I_0^{\mathfrak{R}}(\mathbf{i})$. Now, let us notice that tropical mutations associated to right τ -moves affect only the variables associated to right outlets. Therefore, we get the equality

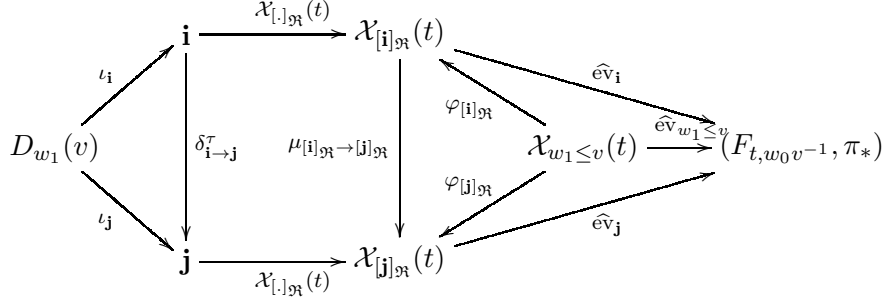
$$\overline{\text{ev}}_{\mathbf{i}}(\mathbf{x}, \mathbf{t}) = \overline{\text{ev}}_{\mathbf{i}'}(\mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{i}']_{\mathfrak{R}}}(\mathbf{x}), \mathbf{t}).$$

Finally, it is clear that the relation $\widehat{\text{ev}}_{\mathbf{i}}(\mathbf{x}) = \overline{\text{ev}}_{\mathbf{i}}(\mathbf{x}, \mathbf{x}(\mathfrak{R}))$ is satisfied for every $\mathbf{i} \in D_{w_1}(v)$. The proposition is then proved, because, by equation (4.5), the cluster variables belonging to $\mathbf{x}(\mathfrak{R})$ are invariant by mutations. \square

For every $t \in H$ and every $w_1 \leq v \in W$, the cluster \mathcal{X} -variety $\mathcal{X}_{w_1 \leq v}(t)$, has been therefore attached to the Poisson submanifold $(F_{t,v}, \pi_*)$. Let us denote $\mathcal{X}_{[\cdot]_{\mathfrak{R}}}(t)$ the application which associates to any double word \mathbf{i} the corresponding seed \mathcal{X} -torus $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}(t)$. Applied to a seed \mathbf{i} , the application $\mathcal{X}_{[\cdot]_{\mathfrak{R}}}(t)$ is therefore the composition of the map \mathcal{X} and the truncation map $\mathbf{t}_{\mathbf{i}_{\mathfrak{R}}(t)}$. We then sum-up Theorem 7.9 and Proposition 7.20 by (abusively) saying that there exists a Poisson map $\widehat{\text{ev}}_{w_1 \leq v} : \mathcal{X}_{w_1 \leq v} \rightarrow (F_{t,v}, \pi_*)$. We thus get the following commutative diagram, which generalize Figure 12. (The link between these truncated cluster varieties $\mathcal{X}_{w_1 \leq v}(t)$, as well as the way to link the different twisted evaluations, is given in the next section via the introduction of saltation maps.)

8. SALTATIONS AND CLUSTER \mathcal{X} -VARIETIES FOR (BB_-, π_*)

We relate the twisted evaluations $\widehat{\text{ev}}_{\mathbf{i}} : \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}} \rightarrow (G, \pi_*)$ of Section 7 by composition of cluster transformations with new birational Poisson isomorphisms called saltations. As a corollary, we get a parametrization of the dual Poisson Lie-group (BB_-, π_*) by a family of cluster \mathcal{X} -varieties; moreover, the cluster \mathcal{X} -varieties of this family are related by saltations indexed by the 1-skeleton of the W -permutohedron P_W .

FIGURE 23. The cluster \mathcal{X} -variety $\mathcal{X}_{w_1 \leq v}(t)$ associated to $(F_{t, w_0 v^{-1}}, \pi_*)$

8.1. Various moves on the set $D(v)$. We sharpen the description of the set $D(v)$ for every $v \in W$. To do that, we enlarge the combinatorics on double words by introducing *dual moves* and mix them with the d^τ -moves described in Section 6. We start by adding a variation of τ -moves, involving the involution $i \mapsto i^*$ on the set $[1, l]$.

Definition 8.1. Let $\mathbf{i} = i_1 \dots i_n$ be a double word. We denote $\mathfrak{R}_{[i_n]}^*(\mathbf{i})$, or simply $\mathfrak{R}^*(\mathbf{i})$ when no confusion occurs, the double word obtained by changing the last letter i of \mathbf{i} into \bar{i}^* :

$$\mathfrak{R}_{i_n}^*(\mathbf{i}) = i_1 \dots i_{n-1} \bar{i}_n^* \quad \text{if } i_n \in [1, l] ;$$

$$\mathfrak{R}_{\bar{i}_n^*}^*(\mathbf{i}) = i_1 \dots i_{n-1} \bar{i}_n^* \quad \text{if } i_n \in [\bar{1}, \bar{l}] .$$

The map $\mathbf{i} \mapsto \mathfrak{R}^*(\mathbf{i})$ is called a *right τ^* -move* on \mathbf{i} . Because the maps $i \mapsto i^*$ and $i \mapsto \bar{i}$ are involutions, it is clear that every map \mathfrak{R}_j^* is an involution on the set of double words. A d^{τ^*} -move is then given by one of these transformations:

- a generalized d -move;
- a τ^* -move.

For every $w \in W$, let $R^{\tau^*}(w)$ be the set of all the double words obtained from a double reduced word $\mathbf{i} \in R(1, w)$ by composition of d^{τ^*} -moves. (The choice of the double word \mathbf{i} doesn't matter, because of Theorem 3.1.)

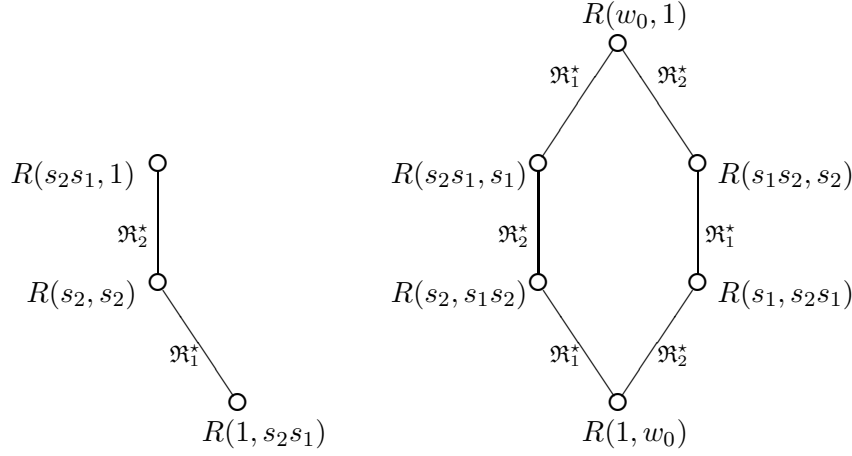
Example 8.2. When $\mathfrak{g} = A_2$, the action of τ^* -move on the double reduced words $\mathbf{i}_1 = 121$, $\mathbf{i}_2 = \bar{2}12$ and $\mathbf{i}_3 = \bar{2}\bar{1}1$ is given by:

$$\mathfrak{R}_1^*(\mathbf{i}_1) = 12\bar{2}, \quad \mathfrak{R}_2^*(\mathbf{i}_2) = \bar{2}1\bar{1}, \quad \text{and} \quad \mathfrak{R}_1^*(\mathbf{i}_3) = \bar{2}\bar{1}\bar{2} .$$

As every set $R^\tau(w)$, the set $R^{\tau^*}(w)$, $w \in W$, is easily described via the W -permutohedron. Indeed, we have the following analog of Lemma 6.4 for the set $R^{\tau^*}(w)$. It is proved in the same way.

Lemma 8.3. *The following statements are satisfied for every $w \in W$.*

- The set $R^{\tau^*}(w)$ is the disjoint union of the labels $R(w'^{\star^{-1}}, ww'^{-1})$ associated to the vertices w' of the W -permutohedron P_W that are crossed or reached by a \uparrow_w -path.
- Two labeled vertices $R(u, v) \subset R^{\tau^*}(w)$ and $R(u', v') \subset R^{\tau^*}(w)$ of P_W are related by the edge s_i if and only if there exist double reduced words $\mathbf{i} \in R(u, v)$ and $\mathbf{j} \in R(u', v')$ such that $\mathbf{j} = \mathfrak{R}^*(\mathbf{i})$.

FIGURE 24. The sets $R^{\tau^*}(s_2 s_1)$ and $R^{\tau^*}(w_0)$ when $\mathfrak{g} = A_2$

Here is the variation of Example 6.5 which has described the sets $R^{\tau}(s_2 s_1)$ and $R^{\tau}(w_0)$ when $\mathfrak{g} = A_2$: Figure 18 is now replaced by Figure 24. We now use the previous τ^* -moves and the involution \square given in Subsection 5.2 to define new moves on double words, besides generalized dn -moves and τ -moves; they are called dual-moves.

Definition 8.4. Let $\mathbf{i} \in R(1, w_0) \cup R(w_0, 1)$ be a positive or negative reduced word associated to w_0 and \mathbf{j} be a double word. The following *dual-move* Δ_j associated to the last letter of \mathbf{j} transforms the product $\mathbf{j}\mathbf{i}$ into the following double word:

$$(8.1) \quad \Delta_j : \mathbf{j}\mathbf{i} \mapsto \mathfrak{R}_j^*(\mathbf{j}) \mathbf{i}^{\square}.$$

Right τ^* -moves and the map \square being involutions, dual-moves are in fact involutions on the set of such products $\mathbf{j}\mathbf{i}$. Let $v \in W$ and $\mathbf{i} \in D(v)$ be a double word. A \hat{d} -move on \mathbf{i} is one of the following transformations. (In particular, every \hat{d} -move is a d^{τ} -move when $v = 1$.)

- a d^{τ} -move;
- a dual-move Δ_i .

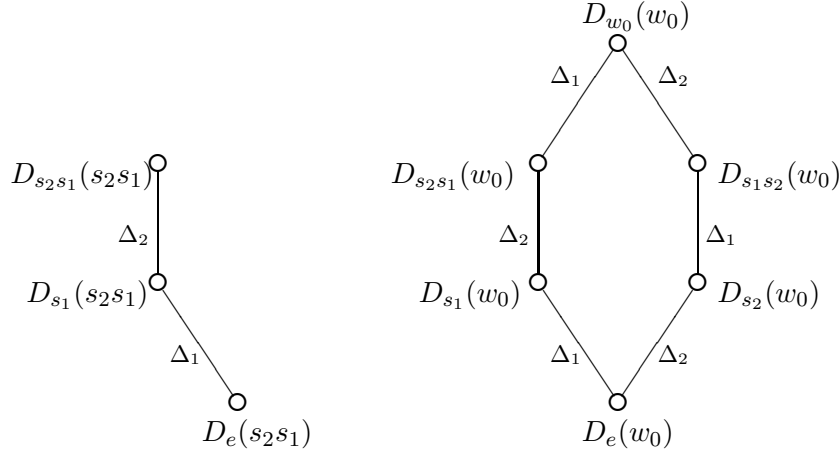
As an example, let us keep the notation of Example 8.2 and consider the double words $\mathbf{j}_1 = \mathbf{i}_1 \mathbf{j}$, $\mathbf{j}_2 = \mathbf{i}_2 \mathbf{j}$ and $\mathbf{j}_3 = \mathbf{i}_3 \mathbf{j}$, with $\mathbf{j} = \overline{121}$. Because the equality $\mathbf{j}^{\square} = 121$ is satisfied, the action of dual moves on these double words give:

$$\Delta_2(\mathbf{j}_1) = 12\overline{2} \ 121, \quad \Delta_1(\mathbf{j}_2) = \overline{2}1\overline{1} \ 121, \quad \text{and} \quad \Delta_2(\mathbf{j}_3) = \overline{2}\overline{1}\overline{2} \ 121.$$

Lemma 8.5. *The following statements are satisfied for every $v \in W$.*

- The set $D(v)$ is the disjoint union of the labels $D_{w_1}(v)$ associated to the vertices w_1 of the W -permutohedron P_W that are crossed or reached by a \uparrow_v -path.
- Two labeled vertices $D_{w_1}(v) \subset D(v)$ and $D_{w'_1}(v) \subset D(v)$ of P_W are related by the edge s_i if and only if there exist trivial double reduced words $\mathbf{i} \in D_{w_1}(v)$ and $\mathbf{j} \in D_{w'_1}(v)$ such that $\mathbf{j} = \Delta_i(\mathbf{i})$.

Proof. The first statement is just a reformulation of the second relation given by (7.5). The second statement is obtained by applying Lemma 7.16 and Lemma 8.3, because of the formula (8.1) describing dual moves. \square

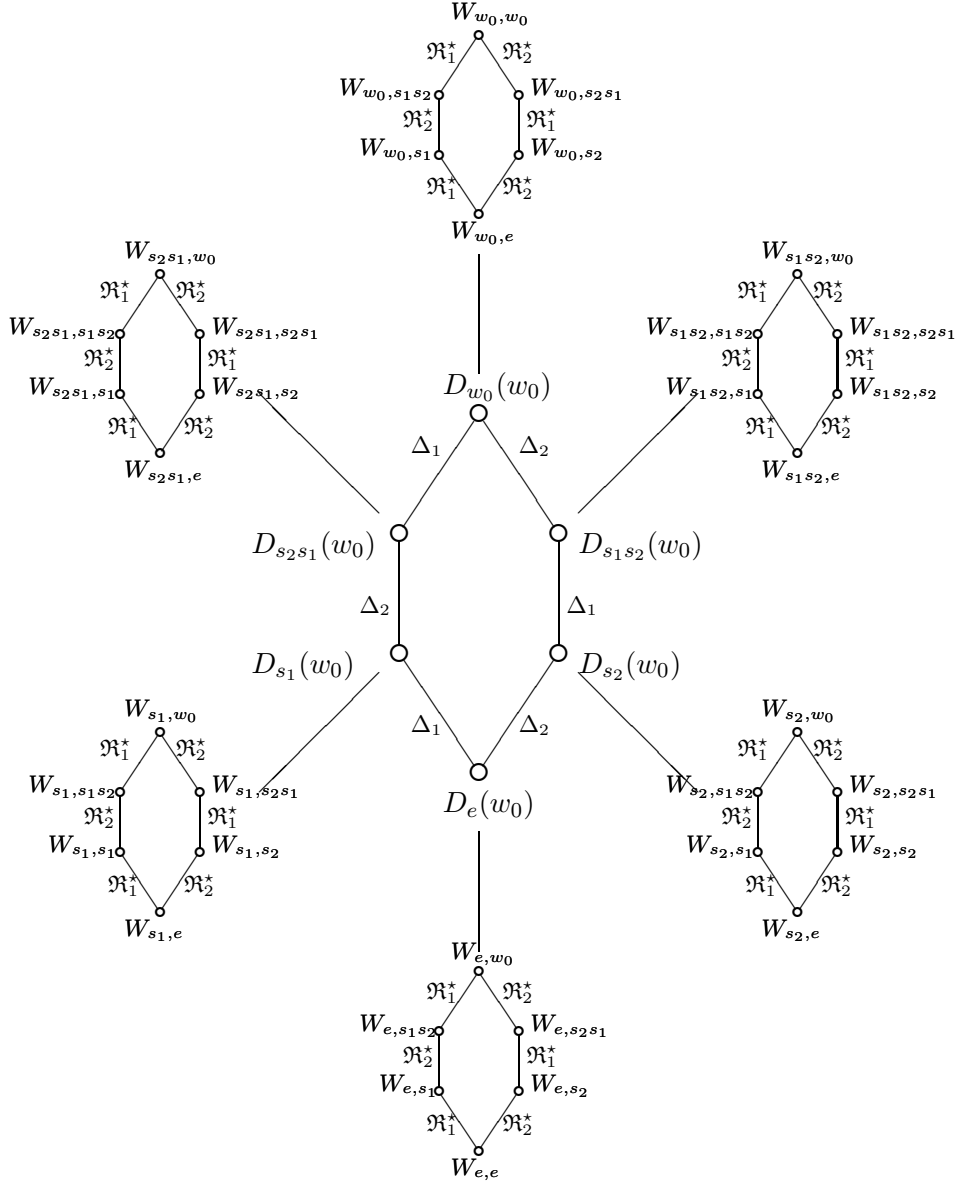
FIGURE 25. The sets $D(s_2s_1)$ and $D(w_0)$ when $\mathfrak{g} = A_2$

We continue our running example with the case $\mathfrak{g} = A_2$. Using Figure 24, we give in Figure 25 the description of the sets $D(s_2s_1)$ and $D(w_0)$ in terms of subsets $D_{w_1}(s_2s_1)$ and $D_{w_1}(w_0)$ related by dual moves. Now, we have seen in Example 7.17 that each subset $D_{w_1}(w_0)$ can be decomposed into sets $W(u, v)$, for appropriate $u, v \in W$, related by right τ^* -moves; this is described by Figure 21. Therefore, mixing Figure 21 and Figure 25, we get in Figure 26 a description of the set $D(w_0)$ as unions of sets $W(u, v)$ related by \hat{d} -moves. Let us notice the double occurring of the permutohedron P_3 in this picture.

8.2. Saltations. We are now ready to introduce the saltations, and use them to describe the cluster combinatorics associated to double words differing from a dual move. Roughly speaking saltations are a generalization of generalized cluster transformations involving truncation maps. When we deal with generalized cluster transformations, the combinatorics giving the formulas is described by the Poisson bivector of the seed \mathcal{X} -torus (i.e. the seed matrix usually denoted ε), which, in turn, is transformed by these generalized cluster transformations. The idea underlying the definition of the saltations is simple: we allow a little more freedom between the combinatorics on seed \mathcal{X} -tori and their Poisson geometry. Let us remember the truncated torus \mathcal{X}_J^0 associated to a truncation map \mathfrak{t}_J and given by Definition 4.4.

Definition 8.6. Let $\mathbf{I} = (I, I_0, \varepsilon, d)$, and $\mathbf{I}' = (I', I'_0, \varepsilon', d')$ be two seeds, $J \subset I$, $J' \subset I'$ be two isomorphic subsets, and $\mathfrak{t}_J(\mathbf{I})$, $\mathfrak{t}_{J'}(\mathbf{I}')$ the related truncation maps. A birational Poisson isomorphism $\Xi : \mathcal{X}_{\mathfrak{t}_J(\mathbf{I})} \rightarrow \mathcal{X}_{\mathfrak{t}_{J'}(\mathbf{I})}$ is said to be a *saltation* (relatively to the subsets J, J') if there exists a generalized cluster transformation $\phi_{\mathbf{I} \rightarrow \mathbf{I}'}^\tau : \mathcal{X}_{\mathbf{I}} \rightarrow \mathcal{X}_{\mathbf{I}'}$ which makes the following diagram commutative for every $\mathbf{t} \in \mathcal{X}_J^0$.

$$(8.2) \quad \begin{array}{ccc} & \xrightarrow{\phi_{\mathbf{I} \rightarrow \mathbf{I}'}^\tau} & \\ \mathcal{X}_{\mathbf{I}} & & \mathcal{X}_{\mathbf{I}'} \\ \mathfrak{t}_J(\mathbf{t}) \downarrow & & \downarrow \mathfrak{t}_{J'}(\mathbf{t}) \\ \mathcal{X}_{\mathfrak{t}_J(\mathbf{I})}(\mathbf{t}) & \xrightarrow{\Xi} & \mathcal{X}_{\mathfrak{t}_{J'}(\mathbf{I}')}(\mathbf{t}) \end{array}$$

FIGURE 26. The set $D(w_0)$ when $\mathfrak{g} = A_2$

Saltation are easily composed: if Ξ_1 is a saltation relatively to the sets J, J' and to a generalized cluster transformation ϕ_1 and Ξ_2 a saltation relatively to the sets J', J'' and to the generalized cluster transformation ϕ_2 , then the composition $\Xi_2 \circ \Xi_1$ is a saltation relatively to the sets J, J'' and to the generalized cluster transformation $\phi_2 \circ \phi_1$.

A few saltations have already been encountered before.

- Every generalized cluster transformation is a saltation: just take the sets J and J' equal to the empty set \emptyset .
- For every $u, v \in W$ and every double reduced words $\mathbf{i}, \mathbf{j} \in R(u, v)$, the cluster transformation $\mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{j}]_{\mathfrak{R}}}$ is the saltation relative to the cluster transformation

$\mu_{\mathbf{i} \rightarrow \mathbf{j}}$, J being the set of right outlets relative to the seed $\mathbf{I}(\mathbf{i})$ and J' the set of right outlets relative to the seed $\mathbf{I}(\mathbf{j})$.

- For every $w \in W$ and every $\mathbf{i} \in R^\tau(w)$, the cluster transformation $\zeta_{[\mathbf{i}]_{\mathfrak{R}}}$ is the saltation relative to the generalized cluster transformation $\zeta_{\mathbf{i}}$, where the set J is the set of right outlets relative to the seed $\mathbf{I}(\mathbf{i})$ and J' is the set of right outlets relative to the seed $\mathbf{I}(\mathbf{i}^\square)$.
- More generally, the saltation associated to any generalized cluster transformation ϕ is always a cluster transformation if the set J contains the directions relative to all the tropical mutations that are used to factorize ϕ , and $\phi(J) \subset J'$.

Remark 8.7. Every generalized cluster transformation $\phi_{\mathbf{I} \rightarrow \mathbf{I}'}$ is a product of symmetries, mutations and tropical mutations, and every symmetry of a finite set J can be decomposed into a product of transpositions. Therefore it is tempting to decompose every saltation Ξ as a product of elementary saltations of the following forms.

$$\begin{array}{ccc}
 \mathcal{X}_{\mathbf{I}} & \xrightarrow{\text{id}} & \mathcal{X}_{\mathbf{I}} \\
 \downarrow \mathfrak{t}_J & & \downarrow \mathfrak{t}_{J'} \\
 \mathcal{X}_{\mathbf{I}_J}(t) & \xrightarrow{\Xi_1} & \mathcal{X}_{\mathbf{I}_{J'}}(t)
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathcal{X}_{\mathbf{I}} & \xrightarrow{\mu_k} & \mathcal{X}_{\mu_k(\mathbf{I})} \\
 \downarrow \mathfrak{t}_J & & \downarrow \mathfrak{t}_J \\
 \mathcal{X}_{\mathbf{I}_J}(t) & \xrightarrow{\Xi_2} & \mathcal{X}_{\mu_k(\mathbf{I})_J}(t)
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{X}_{\mathbf{I}} & \xrightarrow{\mu_k^\tau} & \mathcal{X}_{\mu_k^\tau(\mathbf{I})} \\
 \downarrow \mathfrak{t}_J & & \downarrow \mathfrak{t}_J \\
 \mathcal{X}_{\mathbf{I}_J}(t) & \xrightarrow{\Xi_3} & \mathcal{X}_{\mu_k^\tau(\mathbf{I})_J}(t)
 \end{array}$$

The problem is that the first map Ξ_1 is clearly undefined if the symmetry $s : J \mapsto J'$ is not the identity!

Here is the main reason for introducing saltations. We define for every double reduced word \mathbf{j} , every positive word $\mathbf{i}_+ \in R(1, w_0)$ and every $i \in [1, l]$, the map $\Xi_k : \mathcal{X}_{[\mathbf{j}\mathbf{i}_+\bar{k}]_{\mathfrak{R}}} \rightarrow \mathcal{X}_{[\mathbf{j}\mathbf{i}_+^{\square k^*}]_{\mathfrak{R}}}$ given by

$$(8.3) \quad x_{\Xi_k(i)} = \begin{cases} x_{\zeta_{\mathbf{i}_+}(j)} & \text{if } j < N^i(\mathbf{j}\mathbf{i}_+) ; \\ x_{\zeta_{\mathbf{i}_+}(j)} x_{(N^i(\mathbf{j}\mathbf{i}_+\bar{k}))}^{-1} & \text{if } j = N^i(\mathbf{j}\mathbf{i}_+) < N^i(\mathbf{j}\mathbf{i}_+\bar{k}) ; \\ x_{(j)} & \text{otherwise .} \end{cases}$$

Proposition 8.8. *The map $\Xi_i : \mathcal{X}_{[\mathbf{j}\mathbf{i}_+\bar{i}]_{\mathfrak{R}}} \rightarrow \mathcal{X}_{[\mathbf{j}\mathbf{i}_+^{\square i^*}]_{\mathfrak{R}}}$ is a saltation, but not a generalized cluster transformation.*

Proposition 8.8 will be proved in Subsection 8.4. For the moment, we focus on the link between saltations and dual moves, given by the following result.

Corollary 8.9. *Let $i \in [1, l]$ and \mathbf{i} be a double word such that we can apply the dual move Δ_i on it. Then the following product is a birational Poisson isomorphism.*

$$(8.4) \quad \begin{array}{ccc} \Xi_{s_i} : \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}} & \longrightarrow & \mathcal{X}_{[\Delta_i(\mathbf{i})]_{\mathfrak{R}}} \\ \mathbf{x} & \longmapsto & \mu_{[\mathbf{i}_+^{\square i^*}]_{\mathfrak{R}} \rightarrow [\Delta_i(\mathbf{i})]_{\mathfrak{R}}} \circ \Xi_i \circ \mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{i}_+\bar{i}]_{\mathfrak{R}}}(\mathbf{x}) . \end{array}$$

Proof. We use Proposition 8.8 and the fact that, for every double words \mathbf{j} and \mathbf{k} , the cluster transformation $\mu_{[\mathbf{j}]_{\mathfrak{R}} \rightarrow [\mathbf{k}]_{\mathfrak{R}}}$, when it exists, is a birational Poisson isomorphism between the seed \mathcal{X} -tori $\mathcal{X}_{[\mathbf{j}]_{\mathfrak{R}}}$ and $\mathcal{X}_{[\mathbf{k}]_{\mathfrak{R}}}$. \square

We finally prove that the relations between the various cluster \mathcal{X} -varieties \mathcal{X}_w associated to the different double words $\mathbf{i} \in D(w_0)$ involve saltations and are described by the W -permutohedron P_W .

Lemma 8.10. *Let us replace every label $w' \in W$ of the vertices of the W -permutohedron P_W by the cluster \mathcal{X} -variety \mathcal{X}_w associated to a seed \mathcal{X} -torus $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$ related to a double word $\mathbf{i} \in D_w(w_0)$. We have the following statements.*

- The cluster \mathcal{X} -variety \mathcal{X}_w related to any $w \in W$ contains the seed \mathcal{X} -torus $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$ associated to any double word $\mathbf{i} \in D_w(w_0)$.
- For any $i \in [1, l]$, if two vertices are respectively labeled by \mathcal{X}_w and $\mathcal{X}_{w'}$ of P_W are related by the edge $s_i \in W$, there exist two trivial double words $\mathbf{i}, \mathbf{j} \in D(w_0)$ such that the seed \mathcal{X} -tori associated $\mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$ and $\mathcal{X}_{[\mathbf{j}]_{\mathfrak{R}}}$ are related by the saltation Ξ_{s_i} .

Proof. The first statement is given by Lemma 7.19. And the second statement comes from Lemma 8.5 and Corollary 8.9. \square

8.3. Cluster \mathcal{X} -varieties for (G, π_*) , saltations and the W -permutohedron. We obtain finally, in Theorem 8.12, the cluster combinatorics relating the twisted evaluations of Section 7. This cluster combinatorics involves cluster transformations and saltations.

To any trivial double words $\mathbf{i}, \mathbf{i}' \in D(v)$ such that there exists a \widehat{d} -move $\delta : \mathbf{i} \rightarrow \mathbf{i}'$ we associate a birational Poisson isomorphism $\widehat{\mu}_{\mathbf{i} \rightarrow \mathbf{i}'} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{i}'}$ given by

- the cluster transformation $\mu_{[\mathbf{i}]_{\mathfrak{R}} \rightarrow [\mathbf{i}']_{\mathfrak{R}}}$ if δ is a d^T -move;
- the birational Poisson isomorphism Ξ_{s_i} if δ is the dual-move Δ_i .

From Lemma 8.5, there exists a sequence of \widehat{d} -moves relating any two trivial double words $\mathbf{i}, \mathbf{i}' \in D(v)$. We therefore extend this definition to every $\mathbf{i}, \mathbf{j} \in D(v)$ in the usual way: If \mathbf{i}, \mathbf{j} are trivial double words linked by a sequence of \widehat{d} -moves and $\mathbf{i} \rightarrow \mathbf{i}_1 \rightarrow \dots \rightarrow \mathbf{i}_{n-1} \rightarrow \mathbf{j}$ is the associated chain of elements, we define the map $\widehat{\mu}_{\mathbf{i} \rightarrow \mathbf{j}}$ as the composition $\widehat{\mu}_{\mathbf{i}_{n-1} \rightarrow \mathbf{j}} \circ \dots \circ \widehat{\mu}_{\mathbf{i} \rightarrow \mathbf{i}_1}$. Finally, because every double word $\mathbf{k} \in D(v)$ is related (at least) to a trivial double word of $D(v)$ by a sequence of generalized d -moves, we associate the cluster transformation $\mu_{\mathbf{k} \rightarrow \mathbf{i}}$ to complete the picture. Finally, we get a birational Poisson isomorphism $\widehat{\mu}_{\mathbf{i} \rightarrow \mathbf{i}'} : \mathcal{X}_{\mathbf{i}} \rightarrow \mathcal{X}_{\mathbf{i}'}$ associated to any double words $\mathbf{i}, \mathbf{j} \in D(v)$. We can now relate the Poisson birational isomorphism of Corollary 8.9 with the twisted evaluations of Section 7.

Proposition 8.11. *For every $i \in [\bar{1}, \bar{l}]$ and every double reduced word $\mathbf{i} \in R(s_i, w_0)$ starting with the letter i , we have the following equality.*

$$\widehat{\mathbf{e}}\mathbf{i} = \widehat{\mathbf{e}}\mathbf{v}_{\Delta_i(\mathbf{i})} \circ \Xi_{s_i}.$$

Proposition 8.11 is proved in Subsection 8.5. Now, because of Lemma 8.5, there exists a composition of \widehat{d} -moves δ that satisfy the relation $\delta : \mathbf{i} \rightarrow \mathbf{j}$. Therefore, it suffices to apply Proposition 7.20 and Proposition 8.11 to prove the following result, which was the missing argument to prove Theorem 7.9.

Theorem 8.12. *For every $v \in W$ and $\mathbf{i}, \mathbf{j} \in D(v)$ the maps $\widehat{\mathbf{e}}\mathbf{i}$ and $\widehat{\mathbf{e}}\mathbf{j}$ satisfy the equality $\widehat{\mathbf{e}}\mathbf{i} = \widehat{\mathbf{e}}\mathbf{j} \circ \widehat{\mu}_{\mathbf{i} \rightarrow \mathbf{j}}$.*

Figure 27 describes, in the case $\mathfrak{g} = A_2$, the full picture of cluster combinatorics we obtain for (BB_-, π_*) from the cluster \mathcal{X} -varieties related to (G, π_G) and described in Section 3.

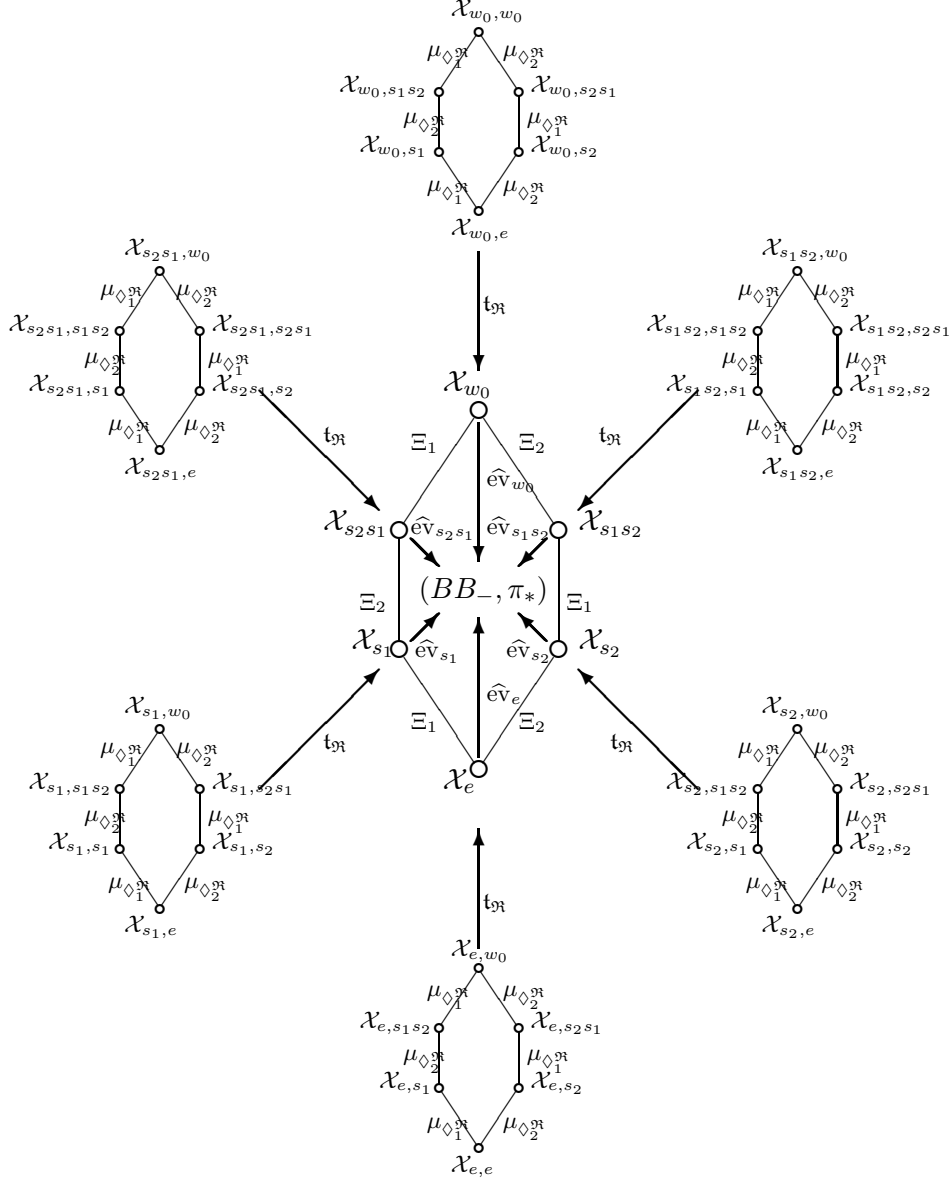


FIGURE 27. Cluster \mathcal{X} -varieties evaluating (BB_-, π_*) when $\mathfrak{g} = A_2$

8.4. Proof of Proposition 8.8. We need a few preliminaries to prove Proposition 8.8. Let us remember the map $\pi_{\mathbf{i}}$ associated to any double \mathbf{i} given by (5.23). Here is a generalization. Let \mathfrak{s} be a \mathcal{X} -split associated to the decomposition $\mathbf{ij} \rightarrow (\mathbf{i}, \mathbf{j})$: for every $\mathbf{x} \in \mathcal{X}_{\mathbf{ij}}$, we define

$$(8.5) \quad equ : \pi_{\mathbf{ij} \rightarrow \mathbf{i}}(\mathbf{x}) = m(\mathbf{x}_{(1)}, \pi_{\mathbf{j}}(\mathbf{x}_{(2)})) \quad \text{and} \quad \pi_{\mathbf{ij} \rightarrow \mathbf{j}}(\mathbf{x}) = m(\pi_{\mathbf{i}}(\mathbf{x}_{(1)}), \mathbf{x}_{(2)}) .$$

(It is clear that these equalities don't depend on the choice of \mathfrak{s} and that they are also satisfied for every $\mathbf{x} \in \mathcal{X}_{[\mathbf{ij}]_{\mathfrak{R}}}$.) Figure 28 and Figure 29 give examples of these maps in the case $\mathfrak{g} = A_3$, whereas the following result is straightforward.

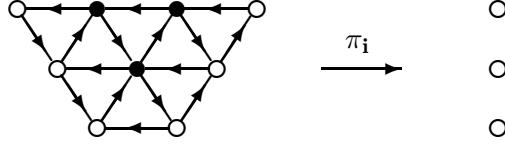


FIGURE 28. The generalized folding $\pi_{\mathbf{i}} : \Gamma_{A_3}(\mathbf{i}) \rightarrow \Gamma_{A_3}(\mathbf{1})$ for $\mathbf{i} = 123121$

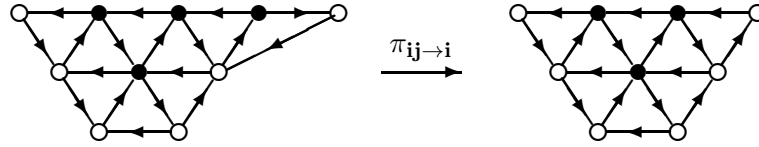


FIGURE 29. The generalized folding $\pi_{\mathbf{ij} \rightarrow \mathbf{i}} : \Gamma_{A_3}(\mathbf{ij}) \rightarrow \Gamma_{A_3}(\mathbf{i})$ for $\mathbf{i} = 123121$ and $\mathbf{j} = \bar{1}$

Lemma 8.13. *Let $u, v \in W$ and $\mathbf{i} \in D(u, 1) \cup D(1, v)$ be a positive or negative double word. The following equality is satisfied for every $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$.*

$$[\text{ev}_{\mathbf{i}}(\mathbf{x})]_0 = \text{ev}_{\mathbf{1}} \circ \pi_{\mathbf{i}}(\mathbf{x}) .$$

Lemma 8.14. *Let $w_1 \leq u, w_2 \leq v \in W$ and $\mathbf{i} \in R(w_2, v)$, $\mathbf{j} \in R(u, w_1)$ be such that $\mathbf{i} = \mathbf{i}_+ \mathbf{i}_-$ and $\mathbf{j} = \mathbf{j}_+ \mathbf{j}_-$. The following equalities are satisfied for every $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$ and $\mathbf{y} \in \mathcal{X}_{\mathbf{j}}$.*

$$[\text{ev}_{\mathbf{i}}(\mathbf{x}) \widehat{v^{-1}}]_{\leq 0} = [\text{ev}_{\mathbf{i}_+} \circ \pi_{\mathbf{i} \rightarrow \mathbf{i}_+}(\mathbf{x}) \widehat{v^{-1}}]_{\leq 0} \quad \text{and} \quad [\widehat{u^{-1}} \text{ev}_{\mathbf{j}}(\mathbf{y})]_{\geq 0} = [\widehat{u^{-1}} \text{ev}_{\mathbf{j}_-} \circ \pi_{\mathbf{j} \rightarrow \mathbf{j}_-}(\mathbf{y})]_{\leq 0} .$$

Proof. We prove the first equality. Let \mathfrak{s} be a \mathcal{X} -split associated to the decomposition $\mathbf{i} \rightarrow (\mathbf{i}_+, \mathbf{i}_-)$. Because $\mathbf{i}_- \in R(w_2, 1)$ and $w_2 \leq v$, the conjugation of $\text{ev}_{\mathbf{i}_-}(\mathbf{x}_{(2)})$ by \widehat{v} belongs to the Borel subgroup B . But it is clear that for every $b \in B$, the equality $[b]_{\leq 0} = [b]_0$ is satisfied. Therefore, we get the result by applying the definition (8.5) for $\pi_{\mathbf{i} \rightarrow \mathbf{i}_+}$ and Lemma 8.13. The second equality is proved in the same way. \square

Lemma 8.15. *Let $w_1 \leq u, w_2 \leq v \in W$ and $\mathbf{i} \in R(w_2, v)$, $\mathbf{j} \in R(u, w_1)$ be double reduced words such that $\mathbf{i} = \mathbf{i}_+ \mathbf{i}_-$ and $\mathbf{j} = \mathbf{j}_+ \mathbf{j}_-$. The following equalities are satisfied:*

$$(8.6) \quad \pi_{\mathbf{i} \rightarrow \mathbf{i}_+} = \zeta_{\mathbf{i}_+}^{-1} \circ \mu_{\mathbf{i}_- \square \rightarrow \mathbf{i}_+ \square} \circ \zeta_{\mathbf{i}_+} \circ \mu_{\mathbf{i} \rightarrow \mathbf{i}_+ \mathbf{i}_+} ;$$

$$\pi_{\mathbf{j} \rightarrow \mathbf{j}_-} = \zeta_{\mathbf{j}_-}^{-1} \circ \mu_{\mathbf{j}_+ \square \rightarrow \mathbf{j}_- \square} \circ \zeta_{\mathbf{j}_-} \circ \mu_{\mathbf{j} \rightarrow \mathbf{j}_- \mathbf{j}_+} .$$

Proof. Let $\mathbf{z} \in \mathcal{X}_{\mathbf{i}_+ \mathbf{i}_+}$ and $b_- := \zeta_{\mathbf{i}_+}^{1, v}(\text{ev}_{\mathbf{i}_+ \mathbf{i}_+}(\mathbf{z}))$. Using Proposition 5.34, equation (3.5), Remark 5.13 and Theorem 3.7, it is clear that the evaluation map $\text{ev}_{\mathbf{i}_+ \square}$ sends the following element on b_-

$$\mu_{\mathbf{i}_- \square \rightarrow \mathbf{i}_+ \square} \circ \zeta_{\mathbf{i}_+}(\mathbf{z}) .$$

But this evaluation also sends the element $\zeta_{\mathbf{i}_+} \circ \pi_{\mathbf{i} \rightarrow \mathbf{i}_+} \circ \mu_{\mathbf{i}_+ \rightarrow \mathbf{i}}(\mathbf{z})$ on b_- , using the first equality of Lemma 8.14, Theorem 3.7 and equation (3.5). Now, the double word \mathbf{i}_+^\square is reduced because \mathbf{i}_+ is a double reduced word. Therefore the maps $\text{ev}_{\mathbf{i}_+^\square}$ and $\zeta_{\mathbf{i}_+}$ are birational isomorphisms and the first equality of (8.6) is proved. The second equality is proved in the same way, using the second equality of Lemma 8.14. \square

Here is a last preliminary. Let us recall that, because of the erasing map in the first line of equation (3.6), the Poisson map $\mu_{\mathbf{i} \rightarrow \mathbf{j}}$ associated to a nil-move $\delta : \mathbf{i} \mapsto \mathbf{j}$ is not a birational isomorphism. We then define the related cluster transformation $\bar{\mu}_{\mathbf{i} \rightarrow \mathbf{j}}$ such that

$$\bar{\mu}_{\mathbf{i} \rightarrow \mathbf{j}} = \varsigma_{\binom{i}{1}} \circ \mu_{\mathbf{i} \rightarrow \mathbf{j}} .$$

Lemma 8.16. *Let $i \in [1, l]$ and $\mathbf{i} \in R(s_i, w_0)$ be a double reduced word starting with the letter \bar{i} . The following equality is satisfied for every $\mathbf{t} \in \mathcal{X}_1$.*

$$\mathbf{t}'_{\mathbf{i}_{\mathfrak{R}}(\mathbf{t})} \circ \bar{\mu}_{\mathbf{i}_+^\square \rightarrow \mathbf{i}_+^\square} \circ \zeta_{\mathbf{i}_+} \circ \mu_{\mathbf{i} \rightarrow \bar{\mathbf{i}}_+} = \Xi_i \circ \mathbf{t}_{\mathbf{i}_{\mathfrak{R}}(\mathbf{t})} ,$$

where $\mathbf{t}'_{\mathbf{i}_{\mathfrak{R}}}$ denotes the truncation map associated to the set $I_0^{\mathfrak{R}}(\mathbf{i})'$ given by

$$(8.7) \quad I_0^{\mathfrak{R}}(\mathbf{i})' = I_0^{\mathfrak{R}}(\mathbf{i}_+^\square i^\star) \cup \left\{ \binom{i}{1} \right\} \setminus \left\{ (N^{i^\star}(\mathbf{i}_+^\square i^\star)) \right\} .$$

Proof. Let us define, for every $i \in [1, l]$ and every double word \mathbf{i} , the erasing map $\varsigma_{\mathfrak{R}^i, \mathbf{i}}$ as the product of j -erasing maps ς_j for every right outlet j which is not a i -vertex:

$$\varsigma_{\mathfrak{R}^i, \mathbf{i}} = \prod_{j \in I_0^{\mathfrak{R}}(\mathbf{i}) \setminus \left\{ (N^i(\mathbf{i})) \right\}} \varsigma_j .$$

Let us then remark that the map Ξ_i can also be defined as

$$(8.8) \quad \Xi_i : \mathcal{X}_{[\mathbf{i}_+ \bar{i}]_{\mathfrak{R}}} \longrightarrow \mathcal{X}_{[\mathbf{i}_+^\square i^\star]_{\mathfrak{R}}} : \mathbf{x} \longmapsto (\zeta_{\mathbf{i}_+} \circ \pi_{\mathbf{i}_+ \bar{i} \rightarrow \mathbf{i}_+}(\mathbf{x}), \mathbf{x}(\mathfrak{R})) .$$

We then use the formulas (8.6) and (8.8) to deduce the following equality, which is satisfied for every $\mathbf{x} \in \mathcal{X}_{[\mathbf{i}_+ \bar{i}]_{\mathfrak{R}}}$.

$$(\varsigma_{\mathfrak{R}^{i^\star}, \mathbf{i}_+^\square} \circ \varsigma_{\binom{i}{1}}) \circ \bar{\mu}_{\mathbf{i}_+^\square \rightarrow \mathbf{i}_+^\square} \circ \zeta_{\mathbf{i}_+} \circ \mu_{\mathbf{i} \rightarrow \bar{\mathbf{i}}_+}(\mathbf{x}, \mathbf{x}(\mathfrak{R})) = \Xi_i \circ \mathbf{t}_{\mathbf{i}_{\mathfrak{R}}(\mathbf{t})}(\mathbf{x}) .$$

It suffices then to use the definition of truncation maps to end the proof of the lemma. \square

Lemma 8.16 therefore implies that the map Ξ_i is a saltation associated to the generalized cluster transformation $\bar{\mu}_{\mathbf{i}_+^\square \rightarrow \mathbf{i}_+^\square} \circ \zeta_{\mathbf{i}_+} \circ \mu_{\mathbf{i} \rightarrow \bar{\mathbf{i}}_+}$, so the first part of Proposition 8.8 is proved.

Lemma 8.17. *Let \mathbf{I} be a seed such that there exist a cluster variable x_i of the seed \mathcal{X} -torus $\mathcal{X}_{\mathbf{I}}$ which is a Casimir function. Then, for every cluster \mathbf{x} of $\mathcal{X}_{\mathbf{I}}$ and every generalized cluster transformation ϕ , we have the equality $x_{\phi(i)} = x_i$ and, for every $j \neq i$, the cluster variable $x_{\phi(j)}$ doesn't depend on the cluster variable x_i .*

Proof. It suffices to factorize the generalized cluster transformation ϕ into a product of tropical mutations, symmetries, and mutations ϕ_k . The properties are then easily checked for each ϕ_k , using formulas for mutations and tropical mutations. \square

To prove that the saltation Ξ_i is not a generalized cluster transformation, it suffices now to apply Lemma 8.17, the second line of formula (8.3), and the fact that every cluster variable x_i , $i \in I_0^{\mathfrak{R}}(\mathbf{i})$ is a Casimir function for every right truncated seed $[\mathbf{i}]_{\mathfrak{R}}$.

8.5. Proof of Proposition 8.11. Let us recall that every element $x \in B_-B$ can be decomposed into $x = [x]_-[x]_0[x]_+$, where $[x]_- \in N_-$, $[x]_0 \in H$, and $[x]_+ \in N$. If $x \in B_-B \cap BB_-$ it can also be decomposed into $x = [[x]]_+[[x]]_0[[x]]_-$, with $[[x]]_- \in N_-$, $[[x]]_0 \in H$, and $[[x]]_+ \in N$. The two decompositions are easily related: using the fact that the map $x \mapsto x^{-1}$ is an involution, we get

$$(8.9) \quad [[x]]_+[[x]]_0[[x]]_- = [x^{-1}]_+^{-1}[x^{-1}]_0^{-1}[x^{-1}]_-^{-1}.$$

Let us now recall the application $\kappa : G \times G \rightarrow G$ given by equation (5.7). Let $t \in H$, $v \in W$ and let us first introduce the map $\Xi_t : L^{v,w_0} \rightarrow L^{w_0,v^{\star-1}}$ given by:

$$(8.10) \quad \Xi_t(gH) = [[gH]]_{\geq 0} \widehat{w_0}]_{\leq 0} [\kappa((t\widehat{w_0})^{-1}, [[gH]]_{< 0})]_+ H.$$

Lemma 8.18. *Let $t \in H$ and $v \in W$. We have for every $g \in G^{v,w_0}$:*

$$\rho_{t,(v,e)_v}(gH) = [\Xi_t(gH)^{\theta} \widehat{w_0}]_{\leq 0}^{\theta} t\widehat{w_0} \Xi_t(gH)^{-1}.$$

Proof. Let us set $b = [[g]]_{\geq 0}$ and $n_- = [[g]]_{< 0}$. We use successively the fact that $[\kappa(\widehat{w_0}^{-1}, n_-)]_0$ and $[\kappa((t\widehat{w_0})^{-1}, n_-)]_0$ are equal (to the unit 1_G of G), the fact that the map $\kappa(g, \cdot)$ commutes with the inverse map $x \mapsto x^{-1}$ for every $g \in G$, and Lemma 7.12 to obtain:

$$\begin{aligned} \rho_{t,(v,e)_v}(gH) &= bn_- t\widehat{w_0} [bn_- \widehat{w_0}]_{\leq 0}^{-1} \\ &= bt\widehat{w_0} \kappa((t\widehat{w_0})^{-1}, n_-^{-1}) ([b\widehat{w_0}]_{\leq 0} [\kappa((\widehat{w_0})^{-1}, n_-^{-1})]_0)^{-1} \\ &= bt\widehat{w_0} ([b\widehat{w_0}]_{\leq 0} [\kappa((t\widehat{w_0})^{-1}, n_-)]_+)^{-1} \\ &= [([b\widehat{w_0}]_{\leq 0} [\kappa((t\widehat{w_0})^{-1}, n_-)]_+)^{\theta} \widehat{w_0}]_{\leq 0}^{\theta} t\widehat{w_0} ([b\widehat{w_0}]_{\leq 0} [\kappa((t\widehat{w_0})^{-1}, n_-)]_+)^{-1} \\ &= [\Xi_t(gH)^{\theta} \widehat{w_0}]_{\leq 0}^{\theta} t\widehat{w_0} \Xi_t(gH)^{-1}. \end{aligned}$$

□

As an immediate corollary, we get, for every $t \in H$ and every $v \in W$:

$$(8.11) \quad \rho_{t,(e,w_0)_v} = \rho_{t,(v,e)_v} \circ \Lambda_t \quad \text{where} \quad \Lambda_t(b_1, cH) = \Xi_t(b_1 cH).$$

Here is now the related cluster combinatorics. We will focus on the case $v = s_i$ because it is all what we need, but similar statements, although more technical, can in fact be obtained for a general $v \in W$. Let us remember the maps $\pi_{\mathbf{i} \rightarrow \mathbf{i}}$ given by (8.5).

Lemma 8.19. *Let $i \in [1, l]$, $t \in H$, and $\mathbf{i} \in R(s_i, w_0)$ be a double reduced word satisfying the equality $\mathbf{i} = \mathbf{i}_+ \mathbf{i}_-$. The following equality is satisfied for every $\mathbf{x} \in \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}(t)$.*

$$[\Xi_t \circ \text{ev}_{\mathbf{i}}(\mathbf{x})]_{\leq 0} = \text{ev}_{\mathbf{i}_+^{\square}} \circ \zeta_{\mathbf{i}_+} \circ \pi_{\mathbf{i} \rightarrow \mathbf{i}_+}(\mathbf{x}).$$

Proof. The double reduced word \mathbf{i} being (s_i, w_0) -adapted, it is easy to see that the equality $[[\text{ev}_{\mathbf{i}}(\mathbf{x})]]_{\geq 0} = \text{ev}_{\mathbf{i}_+} \circ \pi_{\mathbf{i} \rightarrow \mathbf{i}_+}(\mathbf{x})$ is satisfied for every $\mathbf{x} \in \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}(t)$. Therefore, it suffices to apply Theorem 5.37 and (8.10) to prove the lemma. □

Lemma 8.20. *Let $i \in [1, l]$, $g \in G^{s_i, w_0}$ and $t \in H$. Denoting t^{\star} the conjugation of t by $\widehat{w_0}$, we obtain $[\Xi_t(gH)]_+ H = t^{\star} E^{i^{\star}} H$.*

Proof. From $g \in G^{s_i, w_0}$, we get the equality $[[gH]]_{<0} H = F^i H$. So, up to H , we have $[\kappa((t\widehat{w}_0)^{-1}, [[gH]]_{<0})]_+$ equal to $t^* E^{i^*} t^{*-1}$. It suffices then to apply (8.10). \square

Let us remember the reduced evaluations of Subsection 4.1. Because the equality $t = \text{ev}_1(\mathbf{x}(\mathfrak{R}))$ is clear for every $\mathbf{x} \in \mathcal{X}_{[\mathbf{i}]\mathfrak{R}}(t)$, the following proposition is directly implied by using the definitions of Ξ_i and Ξ_t , with Lemma 8.19 and Lemma 8.20, via an amalgamation procedure.

Proposition 8.21. *Let $v \in W$, $t \in H$, and $\mathbf{i} \in R(v, w_0)$ be a double reduced word satisfying the equality $\mathbf{i} = \mathbf{i}_+ \bar{\mathbf{i}}$. The following relation is satisfied for every $\mathbf{x} \in \mathcal{X}_{[\mathbf{i}]\mathfrak{R}}(t)$.*

$$\Xi_t(\text{ev}_{\mathbf{i}}^{\text{red}}(\mathbf{x}^{\text{red}})) = \text{ev}_{\mathbf{i}_+ \bar{\mathbf{i}}^*}^{\text{red}}(\Xi_i(\mathbf{x})^{\text{red}}) .$$

Finally, Proposition 8.11 is deduced from the definition of (w_1, w_2) -maps given in Section 7, equation (8.11) and Proposition 8.21, the properties of the amalgamated product and the definition (8.4) of the birational Poisson isomorphism Ξ_{s_i} .

9. EVALUATIONS AND CLUSTER \mathcal{X} -VARIETIES FOR (G^*, π_{G^*})

We start by giving an alternative way to describe twist maps with mutations and tropical mutations. We then describe the dual Poisson-Lie group (G^*, π_{G^*}) via (w_1, w_2) -maps and provide, in Theorem 9.12, evaluations for (G^*, π_{G^*}) in the spirit of the Kirillov-Reshetikhin multiplicative formula for the quantum R -matrix associated to $\mathcal{U}_q(\mathfrak{g})$. Moreover, birational Poisson isomorphisms using to pass from the positive part to the negative part of (G^*, π_{G^*}) (and vice-versa) can be read on the W -permutohedron: they are described by the \uparrow -paths linking the cluster \mathcal{X} -varieties corresponding to the identity and the longest element w_0 of W .

9.1. Twist maps and coordinates in Schubert cells. We introduce parameterizations of unipotent subgroups of G that will be used to evaluate the Poisson-Lie group (G^*, π_{G^*}) . They involve the generalized cluster transformations of Subsection 5.4. We start by recalling a few facts from [FZ99, Section 2.4]. For every $w \in W$, the corresponding *Schubert cell* $(BwB)/B \subset G/B$ is the image of the Bruhat cell BwB under the natural projection of G onto the flag variety G/B . Let us recall the subgroups $N_+(w) \subset N$ and $N_-(w) \subset N_-$ given by

$$N_+(w) = N \cap \widehat{w} N_- \widehat{w}^{-1} , \quad N_-(w) = N_- \cap \widehat{w}^{-1} N \widehat{w} .$$

The following proposition is essentially well known (cf. [FH91, Corollary 23.60]).

Proposition 9.1. *An element $x \in G$ lies in the Bruhat cell BwB if and only if we have $\widehat{w}^{-1}x \in G_0$ and $[\widehat{w}^{-1}x]_- \in N_-(w)$. Furthermore, the correspondence $\tau_+ : x \mapsto y_+$ given by*

$$y_+ = \tau_+(x) = \widehat{w}[\widehat{w}^{-1}x]_- \widehat{w}^{-1} \in N_+(w)$$

induces a biregular isomorphism between the Schubert cell $(BwB)/B$ and $N_+(w)$.

Let $T : G \rightarrow G : x \mapsto x^T$ be the involutive anti-automorphism of G defined in [FZ99] and given, for every $i \in [1, l]$ and every complex number t , by:

$$a^T = a \ (a \in H), \quad x_i(t)^T = x_{\bar{i}}(t), \quad x_{\bar{i}}(t)^T = x_i(t) .$$

Using the transpose map T , one obtains a counterpart of Proposition 9.1 for the opposite Bruhat cell B_-wB_- .

Proposition 9.2. *An element $x \in G$ lies in B_-wB_- if and only if we have $x\widehat{w}^{-1} \in G_0$ and $[x\widehat{w}^{-1}]_+ \in N_+(w)$. Furthermore, the correspondence $\tau_- : x \mapsto y_-$ given by*

$$y_- = \tau_-(x) = \widehat{w}^{-1}[x\widehat{w}^{-1}]_+\widehat{w} \in N_-(w)$$

induces a biregular isomorphism between the “opposite Schubert cell” $B_- \setminus (B_-wB_-)$ and $N_-(w)$.

The maps τ_+ and τ_- are in fact easily described using mutations and tropical mutations. Let us recall that the group $N_-(w)$ is a unipotent Lie group of dimension $\ell(w)$, hence it is isomorphic to the affine space $\mathbb{C}^{\ell(w)}$ as an algebraic variety. We are going to associate with any negative reduced word $\mathbf{i} = i_1 \dots i_{\ell(w)}$ and every positive reduced word $\mathbf{j} = j_1 \dots j_{\ell(w)}$ the following system of coordinates on $N_{\pm}(w)$ which involves the generalized cluster transformations of equation (5.18). For every $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$ and $\mathbf{y} \in \mathcal{X}_{\mathbf{j}}$, we set

$$(9.1) \quad \begin{aligned} \tau_{\mathbf{i}}(\mathbf{x}) &= \widehat{w}^{-1} \cdot \widehat{s}_{i_1} y_{i_1} (-x_{\binom{i_1}{0}}^{-1}) \cdots \widehat{s}_{i_{\ell(w)}} y_{i_{\ell(w)}} (-x_{\zeta_{\mathbf{i}(\leq \ell(w)-1)} \binom{i_{\ell(w)}}{0}}^{-1}) ; \\ \tau_{\mathbf{j}}(\mathbf{y}) &= x_{j_1} (-y_{\zeta_{\mathbf{j}(\geq 2)} \binom{j_1}{N^{j_1}(\mathbf{j})}}^{-1}) \widehat{s}_{j_1} \cdots x_{j_{\ell(w)}} (-y_{\binom{j_{\ell(w)}}{N^{j_{\ell(w)}}(\mathbf{j})}}^{-1}) \widehat{s}_{j_{\ell(w)}} \cdot \widehat{w}^{-1} . \end{aligned}$$

Now, for every w , every reduced word $\mathbf{i} = i_1 \dots i_{\ell(w)} \in R(w)$, and every $k \in [1, \ell(w)]$, let us set $w_{\mathbf{i}_{>k}} := s_{i_{k+1}} \dots s_{i_n}$ and $w_{\mathbf{i}_{<k}} := s_{i_1} \dots s_{i_{k-1}}$. For every $u \in W$, $t \in \mathbb{C}$ and $i \in [1, l]$, we denote:

$$(9.2) \quad x_{u(i)}(t) := \widehat{u}^{-1} x_i(t) \widehat{u} \quad \text{and} \quad y_{u(i)}(t) := \widehat{u}^{-1} y_i(t) \widehat{u} .$$

It is well-known, and straightforward to check via an induction over the length on W , that the following equalities are satisfied for every w and every complex numbers $t_1, \dots, t_{\ell(w)}$.

$$(9.3) \quad \begin{aligned} \prod_{k=1}^{\ell(w)} y_{w_{\mathbf{i}_{>k}}(i_k)}(t_k) &= \widehat{w}^{-1} \cdot \prod_{k=1}^{\ell(w)} \widehat{s}_{i_k} y_{i_k}(t_k) \\ \text{and} \\ \prod_{k=1}^{\ell(w)} x_{w_{\mathbf{i}_{<k}}(i_k)}(t_k) &= \prod_{k=1}^{\ell(w)} x_{i_k}(t_k) \widehat{s}_{i_k} \cdot \widehat{w}^{-1} . \end{aligned}$$

Lemma 9.3. *For every $w \in W$, $\mathbf{i} \in R(w, 1)$, $\mathbf{j} \in R(1, w)$ and $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$, $\mathbf{y} \in \mathcal{X}_{\mathbf{j}}$ we have*

$$[\widehat{w}^{-1} \text{ev}_{\mathbf{i}}(\mathbf{x})]_- = \tau_{\mathbf{i}}(\mathbf{x}) \quad \text{and} \quad [\text{ev}_{\mathbf{j}}(\mathbf{y})\widehat{w}]_+ = \tau_{\mathbf{j}}(\mathbf{y}) .$$

Proof. Let us focus on the first relation. Using equation (5.12) and the negative projection $[\cdot]_- : B_-B \rightarrow N_- : x \mapsto [x]_-$ on the unipotent subgroup $N_- \subset G$ associated to the Gauss decomposition (4.9), we are led to the equality

$$[\widehat{w_{\mathbf{i}_{>k-1}}}^{-1} \text{ev}_{\mathbf{i}(k-1)} \circ \zeta_{\mathbf{i}(\leq k-1)}(\mathbf{x})]_- = y_{w_{\mathbf{i}_{>k}}(i_k)} (-x_{\zeta_{\mathbf{i}(\leq k-1)} \binom{i_k}{0}}^{-1}) [\widehat{w_{>k}}^{-1} \text{ev}_{\mathbf{i}(k)} \circ \zeta_{\mathbf{i}(\leq k)}(\mathbf{x})]_- .$$

The first relation is then obtained by iteration of this formula because of the first equality of (9.3). The second relation is proved in the same way, using the second equality of (9.3). \square

Proposition 9.4. *For every $w \in W$, $\mathbf{i} \in R(w, 1)$, $\mathbf{j} \in R(1, w)$ and $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$, $\mathbf{y} \in \mathcal{X}_{\mathbf{j}}$, we have the equalities*

$$\tau_-(\text{ev}_{\mathbf{i}}(\mathbf{x})) = \widehat{w} \tau_{\mathbf{i}}(\mathbf{x}) \widehat{w}^{-1} \quad \text{and} \quad \tau_+(\text{ev}_{\mathbf{j}}(\mathbf{y})) = \widehat{w}^{-1} \tau_{\mathbf{j}}(\mathbf{y}) \widehat{w} .$$

Proof. Simply deduced from the preceding lemma and the expression of τ_- and τ_+ given by Proposition 9.1 and Proposition 9.2. \square

9.2. From (G_0, π_*) to (G^*, π_{G^*}) via $(w_1, w_2)_{w_0}$ -maps. Here are a few preliminary maps to get the evaluation maps related to (G^*, π_{G^*}) which are given in the next subsection. Let us remember the variation of the Gauss decomposition given by the formula (8.9).

Lemma 9.5. *The following equalities are satisfied for every $w_2 \in W$ and every $t \in H$.*

$$[[\rho_{t,(e,w_2)_{w_0}}(b_1, gH)]]_- = [b_1 \widehat{w_0}^{-1}]_- \quad \text{and} \quad [[\rho_{t,(w_0,w_2)_{w_0}}(c_1, gH)]]_+ = [(c_1 \widehat{w_0})^{-1}]_+^{-1}.$$

Proof. We use definitions of $(w_1, w_2)_v$ -maps, equation (8.9), and the fact that conjugating every element of the Borel subgroup B_\pm by $\widehat{w_0}$ gives an element of the opposite Borel subgroup B_\mp to deduce

$$(9.4) \quad \begin{aligned} \rho_{t,(e,e)_{w_0}}(b_1, bH) &= \underbrace{b_1 b \, t \widehat{w_0} [b \widehat{w_0}]_{\leq 0}^{-1} \widehat{w_0}^{-1} [b_1 \widehat{w_0}^{-1}]_{\geq 0}^{-1}}_{\in B} \underbrace{[b_1 \widehat{w_0}^{-1}]_+^{-1}}_{\in N_-}; \\ \text{and} \\ \rho_{t,(w_0,e)_{w_0}}(c_1, bH) &= \underbrace{[[c_1 \widehat{w_0}]]_+}_{\in N_+} \underbrace{[[c_1 \widehat{w_0}]]_{\leq 0} \widehat{w_0}^{-1} b t \widehat{w_0} (c_1 [b \widehat{w_0}]_{\leq 0})^{-1}}_{\in B_-}. \end{aligned}$$

Moreover, it is clear that the unipotent part in the previous equalities doesn't depend on the element $bH \in N$ relative to the choice of w_2 . Therefore the lemma is true for every $w_2 \in W$. \square

Let us recall the map Λ_t given by equation (8.11).

Proposition 9.6. *Let $b_1, b \in B$ and $t \in H$. The triplet $(h, n, n_-) \in H \times N \times N_-$, such that the equality*

$$n h n_-^{-1} = \rho_{t,(e,e)_{w_0}}(b_1, bH)$$

is satisfied, is given by the following formulas.

$$(9.5) \quad \begin{cases} h &= [b_1 b]_0 \, t \widehat{w_0} [b \widehat{w_0}]_0^{-1} \widehat{w_0}^{-1} [b_1 \widehat{w_0}^{-1}]_0^{-1}; \\ n &= [(\Lambda_t(b_1, \zeta^{1,w_0}(b)H) \, \widehat{w_0})^{-1}]_+^{-1}; \\ n_- &= [b_1 \widehat{w_0}^{-1}]_-. \end{cases}$$

Proof. The negative unipotent part of (9.5) is given by the left equality of Lemma 9.5, its diagonal part is given by the first equation of (9.4), and its positive unipotent part is given by Theorem 5.37 and Lemma 8.11 and the right equality of Lemma 9.5. \square

9.3. Evaluations related to (G^*, π_{G^*}) . We give the cluster combinatorics on (G^*, π_{G^*}) . To do that, we start by giving evaluation maps for the elements $h \in H$ in (9.5).

Proposition 9.7. *The equality $[[\widehat{\mathbf{ev}}_1(\mathbf{x})]]_0 = \mathbf{ev}_1(\mathbf{X})$ is satisfied for every double word $\mathbf{i} = \mathbf{i}_1 \mathbf{i}_2 \in D_e(w_0)$ and every cluster $\mathbf{x} \in \mathcal{X}_{[\mathbf{i}]}^{\text{pr}}$ if and only if $\mathbf{X} = (X_1, \dots, X_l)$ is the set of monomials given by*

$$X_i = x_{\binom{i}{N^i(\mathbf{i})}} \prod_{k=1}^l \prod_{\ell=1}^{\ell(w_0)} \prod_{j_1 < N^i \ell(\mathbf{i}_1 \ell), j_2 < N^i \ell(\mathbf{i}_2 \ell)} (-x_{\binom{i_\ell^*}{j_1}} x_{\binom{i_\ell^*}{N^i \ell(\mathbf{i}_1 \ell) + j_2}})^{(A^{-1})_{ki} \langle \alpha_{i_\ell}^\vee, w_0 \geq_\ell^{-1} \omega_k \rangle}.$$

Because of its length, the proof of Proposition 9.7 is postponed to Subsection 9.4. Let us however stress that the same kind of monomial formula would have been obtained by choosing a trivial double word $\mathbf{i} = \mathbf{i}_1 \mathbf{i}_2 \in D_{w_0}(w_0)$.

Example 9.8. As usual let us set $\mathfrak{g} = A_2$ and take $t \in H$. We consider a cluster $\mathbf{x} \in \mathcal{X}_{[121 \ 121]_{\mathfrak{R}}}(t)$ and the related elements

$$b_1 = \text{ev}_{121}^{\text{red}}(x_{(0)}^{(1)}, x_{(1)}^{(1)}, x_{(0)}^{(2)}), \quad b = \text{ev}_{121}^{\text{red}}(x_{(2)}^{(1)}, x_{(3)}^{(1)}, x_{(1)}^{(2)}) \quad \text{and} \quad t = \text{ev}_1(t_1, t_2).$$

Using Example 5.33, Lemma 5.15, Lemma 5.16, Proposition 5.17, and equation (9.5), or simply the formula above, we get

$$\begin{aligned} [[\widehat{\text{ev}}_{121 \ 121}(\mathbf{x})]]_0 &= \text{ev}_1 \circ \mathbf{m}((x_{(0)}^{-1} x_{(1)}^{(1)}, x_{(2)}^{-1} x_{(1)}^{-1}), (x_{(1)}^{(2)} x_{(3)}^{(1)}, x_{(2)}^{(1)} x_{(3)}^{-1}), (t_1, t_2)) \\ &= \text{ev}_1(x_{(0)}^{-1} x_{(1)}^{(1)} x_{(3)}^{(1)} x_{(2)}^{(2)} t_1, x_{(2)}^{-1} x_{(1)}^{-1} x_{(2)}^{(1)} x_{(3)}^{-1} t_2). \end{aligned}$$

We then focus on the evaluation maps relative to the elements $n \in N$ and $n_- \in N_-$ in equation (9.5). Let us remember the involutions \star and \circ on double words and seed \mathcal{X} -tori given by Subsection 5.2.

Lemma 9.9. *For every $\mathbf{i} \in R(1, w_0)$, $\mathbf{j} \in R(w_0, 1)$ and $\mathbf{x} \in \mathcal{X}_{\mathbf{i}}$, $\mathbf{y} \in \mathcal{X}_{\mathbf{j}}$ we have*

$$[\text{ev}_{\mathbf{i}}(\mathbf{x}) \widehat{w_0}^{-1}]_- = \tau_{\mathbf{i}^\star}(\mathbf{x}^\star) \quad \text{and} \quad [[\text{ev}_{\mathbf{j}}(\mathbf{y}) \widehat{w_0}]]_+^{-1} = \tau_{\mathbf{j}^\circ}(\mathbf{y}^\circ).$$

Proof. The first relation comes from Lemma 5.15 and Lemma 9.3. Equation (8.9) and Corollary 5.17 implies that the L.H.S. of the second relation is equal to $[\text{ev}_{\mathbf{j}^\circ}(\mathbf{y}^\circ) \widehat{w_0}]_+$. Lemma 9.3 and Lemma 5.16 then lead to the R.H.S. of the second relation. \square

Lemma 9.10. *Let $\mathbf{i}, \mathbf{j} \in D(w_0)$, \mathbf{i} be a $(e, e)_{w_0}$ -word, \mathbf{j} be a $(w_0, e)_{w_0}$ -word, and \mathbf{s}, \mathbf{s}' be \mathcal{X} -splits relative respectively to the $(w_1, w_2)_{w_0}$ -decompositions $\mathbf{i} \rightarrow (\mathbf{i}_1, \mathbf{i}_2)$, and $\mathbf{j} \rightarrow (\mathbf{j}_1, \mathbf{j}_2)$. The following equalities are satisfied.*

$$[[\widehat{\text{ev}}_{\mathbf{i}}(\mathbf{x})]]_-^{-1} = \tau_{\mathbf{i}_1^\star}(\mathbf{x}_{(1)}^\star) \quad \text{and} \quad [[\widehat{\text{ev}}_{\mathbf{j}}(\mathbf{y})]]_+^{-1} = \tau_{\mathbf{j}_1^\circ}(\mathbf{y}_{(1)}^\circ).$$

Proof. The relations are derived from Lemma 7.8, Lemma 9.5 and Lemma 9.9. \square

We can then get the cluster combinatorics on (G^*, π_{G^*}) . Let us associate to any double word $\mathbf{i} \in D(w_0)$ some double words $\mathbf{i}_e \in D_e(w_0)$ and $\mathbf{i}_{w_0} \in D_{w_0}(w_0)$ being respectively a (e, w_0) -trivial double word and a (w_0, e) -trivial double word. Moreover, let us respectively denote $\mathbf{i}_{e+} \in R(1, w_0)$ and $\mathbf{i}_{w_0-} \in R(w_0, 1)$, the positive part of \mathbf{i}_e and the negative part of \mathbf{i}_{w_0} , accordingly the definition of Subsection 5.4.3, and denote $\wp_e : \mathcal{X}_{[\mathbf{i}_e]_{\mathfrak{R}}} \rightarrow \mathcal{X}_{\mathbf{i}_{e+}}^{\text{red}}$ and $\wp_{w_0} : \mathcal{X}_{[\mathbf{i}_{w_0}]_{\mathfrak{R}}} \rightarrow \mathcal{X}_{\mathbf{i}_{w_0-}}^{\text{red}}$ the corresponding canonical projections on seed \mathcal{X} -tori. Let us finally remember the birational Poisson isomorphisms $\widehat{\mu}_{\mathbf{i} \rightarrow \mathbf{i}_e}$ and $\widehat{\mu}_{\mathbf{i} \rightarrow \mathbf{i}_{w_0}}$ defined in Subsection 8.3. To any $\mathbf{x} \in \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$, we associate the clusters

$$\mathbf{x}_e = \wp_e \circ \widehat{\mu}_{\mathbf{i} \rightarrow \mathbf{i}_e}(\mathbf{x}) \quad \text{and} \quad \mathbf{x}_{w_0} = \wp_{w_0} \circ \widehat{\mu}_{\mathbf{i} \rightarrow \mathbf{i}_{w_0}}(\mathbf{x}),$$

and derive the following lemma from Proposition 9.7, Lemma 9.10 and Theorem 8.12.

Lemma 9.11. *The following decomposition is satisfied for every $w \in W$, every double word $\mathbf{i} \in D_w(w_0)$ and every $\mathbf{x} \in \mathcal{X}_w$.*

$$\begin{cases} [[\widehat{\text{ev}}_{\mathbf{i}}(\mathbf{x})]]_0 &= \text{ev}_{\mathbf{1}}(\mathbf{X}_e) ; \\ [[\widehat{\text{ev}}_{\mathbf{i}}(\mathbf{x})]]_+ &= \tau_{\mathbf{i}_{w_0}}(\mathbf{x}_{w_0}^\circ)^{-1} ; \\ [[\widehat{\text{ev}}_{\mathbf{i}}(\mathbf{x})]]_-^{-1} &= \tau_{\mathbf{i}_e^*}(\mathbf{x}_e^*) ; \end{cases}$$

where $\mathbf{X}_e = (X_1, \dots, X_l)$ is the set of monomials given in Proposition 9.7 applied to the cluster $\widehat{\mu}_{\mathbf{i} \rightarrow \mathbf{i}_e}(\mathbf{x})$.

Now, let us recall that the map $\phi : (G^*, \pi_{G^*}) \rightarrow (BB_-, \pi_*)$ given by the formula $(nh, n_-h^{-1}) \mapsto nh^2n_-^{-1}$ is not an isomorphism but a covering of degree 2^l . Example 9.8 in particular shows that in the general case we cannot expect to directly obtain rational evaluations for the dual Poisson-Lie group (G^*, π_{G^*}) , because of this covering $h \mapsto h^2$ on the Cartan subgroup H of G . The remedying idea is to take covers on cluster variables which mimic ϕ . Let $\mathbf{I} = (I, I_0, \varepsilon, d)$ be a seed; the seed \mathcal{X} -torus denoted $\mathcal{X}_{\mathbf{I}/2}$ is the torus $(\mathbb{C}_{\neq 0})^{|I|}$ given with the Poisson bracket

$$\{x_i, x_j\} = \frac{\widehat{\varepsilon}_{ij}}{4} x_i x_j ,$$

where $\{x_i \mid i \in I\}$ still denote the standard coordinates on the factors. In particular, the following map is a Poisson covering of degree $2^{|I|}$.

$$(9.6) \quad \mathbf{c}_{\mathcal{X}} : \mathcal{X}_{\mathbf{I}/2} \longrightarrow \mathcal{X}_{\mathbf{I}} : (x_1, \dots, x_{|I|}) \longmapsto (x_1^2, \dots, x_{|I|}^2) .$$

Thus, Lemma 9.11 and the fact that the maps $\mathbf{c}_{\mathcal{X}}$ and ϕ are Poisson covering whose degrees are some powers of 2 lead us to the following result.

Theorem 9.12. *Let $\mathbf{i} \in D(w_0)$. The following evaluation map $\text{Ev}_{\mathbf{i}}$ is a Poisson covering of degree 2^n , for some $n \leq \dim G$, onto a Zarisky open set of G^* .*

$$\text{Ev}_{\mathbf{i}} : \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}^{1/2} \rightarrow (G^*, \pi_{G^*}) : \mathbf{x} \mapsto (\text{ev}_{\mathbf{i}}^+(\mathbf{x}), \text{ev}_{\mathbf{i}}^-(\mathbf{x}))$$

$$(9.7) \quad \text{where} \quad \begin{cases} \text{ev}_{\mathbf{i}}^+(\mathbf{x}) &= \tau_{\mathbf{i}_{w_0}}(\mathbf{c}_{\mathcal{X}}(\mathbf{x})_{w_0}^\circ)^{-1} \text{ev}_{\mathbf{1}}(\mathbf{X}_e) ; \\ \text{ev}_{\mathbf{i}}^-(\mathbf{x}) &= \tau_{\mathbf{i}_e^*}(\mathbf{c}_{\mathcal{X}}(\mathbf{x})_e^*) \text{ev}_{\mathbf{1}}(\mathbf{X}_e)^{-1} , \end{cases}$$

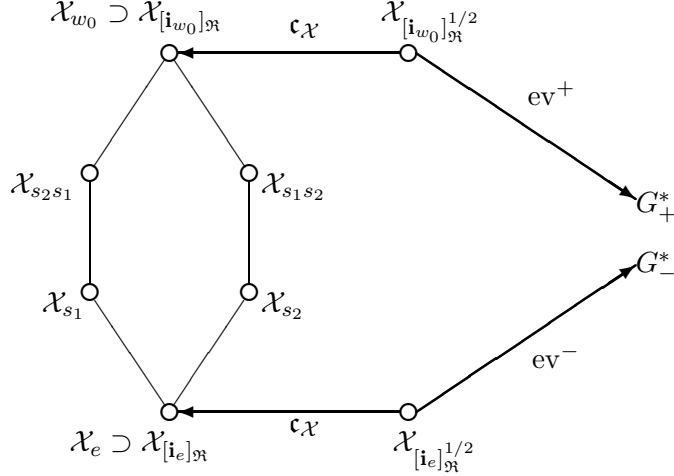
and the set $\mathbf{X}_e = (X_1, \dots, X_l)$ is the same as in Lemma 9.11.

Remark 9.13. A careful study of the cluster variables appearing in the monomial formulas describing $\mathbf{X}_e = (X_1, \dots, X_l)$ and given in Proposition 9.7 leads to a choice of a subcovering of the covering $\mathbf{c}_{\mathcal{X}}$ that minimizes the value n of the previous theorem.

Example 9.14. When the equality $\mathfrak{g} = A_2$ is satisfied, the heuristics of Theorem 9.12 is illustrated by Figure 30, where we have used the notation $G^* = (G_+^*, G_-^*)$ to abbreviate the description (2.10). In particular, if we choose the double word $\mathbf{i} = 121121$, then we can take $\mathbf{i}_e = \mathbf{i}$ and $\mathbf{i}_{w_0} = \overline{21}\overline{21}21$. Therefore, to any $\mathbf{x} \in \mathcal{X}_{[\mathbf{i}]_{\mathfrak{R}}}$ are associated the elements:

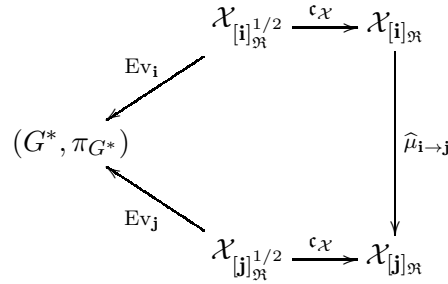
$$\mathbf{x}_e = \wp_e(\mathbf{x}) \quad \text{and} \quad \mathbf{x}_{w_0} = \wp_{w_0} \circ \Xi_{s_1} \circ \mu_{\overline{21}\overline{1}121 \rightarrow \overline{21}\overline{1}121} \circ \Xi_{s_2} \circ \mu_{12\overline{2}121 \rightarrow \overline{2}12121} \circ \Xi_{s_1}(\mathbf{x}),$$

whereas \mathbf{X}_e has already been given in Example 9.8.

FIGURE 30. Evaluations related to (G^*, π_{G^*}) when $\mathfrak{g} = A_2$

These evaluation maps on (G^*, π_{G^*}) are, of course, compatible with the cluster combinatorics already developed. The following theorem is then derived from Theorem 8.12 and the definitions (9.6) and (9.7) of the covering $\mathfrak{c}\mathcal{X}$ and the evaluation map Ev_i .

Theorem 9.15. *The following diagram is commutative for any double words $\mathbf{i}, \mathbf{j} \in D(w_0)$.*



9.4. Proof of Proposition 9.7. The main ingredient to prove Proposition 9.7 is the factorization theorem [FZ99, Theorems 1.10 and formula (1.21)] of Fomin and Zelevinsky. Here, we mainly follow the exposition of [KZ02]. Let \tilde{G} be the simply connected cover of G , and denote \tilde{B} , \tilde{B}_- the Borel subgroups of \tilde{G} such that their images in G are respectively B and B_- , and the intersection $\tilde{H} = \tilde{B} \cap \tilde{B}_-$. In the same way, for every $u, v \in W$, denote $\tilde{G}^{u,v}$ the double Bruhat cell in \tilde{G} whose image in G is $G^{u,v}$. For $x \in \tilde{B}\tilde{B}_-$ and a fundamental weight ω_i , define $\overline{\Delta}_i(x) = [x]_0^{\omega_i}$. It is shown in [FZ99] that $\overline{\Delta}_i$ extends to a regular function on \tilde{G} . For type A_n (when $\tilde{G} = SL(n+1, \mathbb{C})$), this is just the principal $i \times i$ minor of a matrix x . For any pair $u, v \in W$, the corresponding *generalized minor* is a regular function on \tilde{G} given by

$$\Delta_{u\omega_i, v\omega_i}(x) = \overline{\Delta}_i(\hat{u}^{-1}x\hat{v}).$$

It is shown in [FZ99] that these functions are well defined, that is they depend only on the weights $u\omega_i$ and $v\omega_i$ and do not depend on the particular choice of u and v . For $i = 1, \dots, l$, we denote $\varepsilon(i) = +1$ and $\varepsilon(\bar{i}) = -1$, and recall that $|i| = |\bar{i}| = i$. In what follows, we fix $u, v \in W$ and a double reduced word \mathbf{i} of (u, v) . We append l entries i_{m+1}, \dots, i_{m+l} to \mathbf{i}

by setting $i_{m+j} = \bar{j}$. For $k = 1, \dots, m$, we set

$$u_{\geq k} = \prod_{\substack{\ell=m, \dots, k \\ \varepsilon(i_\ell)=-1}} s_{|i_\ell|}, \quad v_{< k} = \prod_{\substack{\ell=1, \dots, k-1 \\ \varepsilon(i_\ell)=+1}} s_{|i_\ell|},$$

where the notation implies that the index ℓ in the first (resp. second) product is decreasing (resp. increasing). We also set $u_{\geq k} = e$, $v_{< k} = v$ for $k = m+1, \dots, m+l$. For example, if $\mathbf{i} = 1\bar{2}\bar{2}3\bar{3}2\bar{1}$ then $u_{\geq 4} = s_1 s_3$, $v_{< 4} = s_1 s_2$. For every $k = 1, \dots, m+l$, we set $\gamma^k = u_{\geq k} \omega_{|i_k|}$, $\delta^k = v_{< k} \omega_{|i_k|}$, and introduce a regular function M_k on $\tilde{G}^{u,v}$ by setting

$$(9.8) \quad M_k(x) = \Delta_{\gamma^k, \delta^k}(x'),$$

where x' is the twist of x given by the formula (5.14). We refer to the family M_1, \dots, M_{m+l} as *twisted minors* associated with a reduced word \mathbf{i} . Their significance stems from the following result (see [FZ99, Theorems 1.2, 1.9, 1.10 and formula (1.21)]). Let us remember the notation given in (5.10) and let us define, for every double word $\mathbf{i} = i_1 \dots i_m$, the map $x_{\mathbf{i}} : \mathbb{C}_{\neq 0}^m \rightarrow G$ by

$$x_{\mathbf{i}}(\mathbf{t}) = x_{i_1}(t_1) \cdots x_{i_m}(t_m) \quad \text{where} \quad \mathbf{t} = (t_1, \dots, t_m).$$

Theorem 9.16. [KZ02, Theorem 2.3] *The map $x_{\mathbf{i}} : \tilde{H} \times \mathbb{C}^m \rightarrow \tilde{G}$ given by*

$$x_{\mathbf{i}}(a; t_1, \dots, t_m) = a x_{i_1}(t_1) \cdots x_{i_m}(t_m)$$

restricts to a biregular isomorphism between a complex torus $\tilde{H} \times (\mathbb{C} - \{0\})^m$ and a Zariski open subset $U_{\mathbf{i}} = \{x \in \tilde{G}^{u,v} : M_k(x) \neq 0 \text{ for } 1 \leq k \leq m+l\}$ of the double Bruhat cell $\tilde{G}^{u,v}$. Furthermore, for $k = 1, \dots, m+l$ and $x = x_{\mathbf{i}}(a; t_1, \dots, t_m) \in U_{\mathbf{i}}$, we have

$$(9.9) \quad M_k(x) = a^{-u\gamma^k} \prod_{\substack{1 \leq \ell < k \\ \varepsilon(i_\ell)=-1}} t_\ell^{\langle \alpha_{|i_\ell|}^\vee, u_{\geq \ell}^{-1} \gamma^k \rangle} \prod_{\substack{k \leq \ell \leq m \\ \varepsilon(i_\ell)=+1}} t_\ell^{\langle \alpha_{|i_\ell|}^\vee, v_{< \ell+1}^{-1} \delta^k \rangle}.$$

We are going to use this theorem to prove Proposition 9.7. Let us first remark that for $u = w_0$ and $v = e$, we have the equalities $\gamma^k = \omega_k$ and $\delta^k = \omega_k$ for every $k > \ell(w_0)$. Now, let us notice that if b_1 and b belong to the double Bruhat cell \tilde{G}^{e, w_0} , then the elements

$$b_1^\star = \widehat{w_0}^{-1} b_1 \widehat{w_0} \quad \text{and} \quad b^\circ = \widehat{w_0}^{-1} b^{-1} \widehat{w_0}$$

belong to the double Bruhat cell $\tilde{G}^{w_0, e}$. Therefore, using the definition of the twist map $x \mapsto x'$, the formula (9.8), and the fact that the relation $a^\theta = a^{-1}$ is satisfied for every $a \in \tilde{H}$, we get the following equalities for every $k > \ell(w_0)$.

$$M_k(b_1^\star) = ([\widehat{w_0}^{-1} b_1^\star]_{\geq 0}^\theta)^{\omega_{i_k}} = [b_1 \widehat{w_0}^{-1}]_0^{-\omega_k} \quad \text{and} \quad M_k(b^\circ)^{-1} = (\widehat{w_0} [b \widehat{w_0}]_0^{-1} \widehat{w_0}^{-1})^{\omega_k}.$$

Taking any $\mathbf{i} \in R(w_0, 1)$ to parameterize $b_1^\star = a x_{\mathbf{i}}(t_1^\star, \dots, t_{\ell(w_0)}^\star)$ and $b^\circ = a' x_{\mathbf{i}}(t_1^\circ, \dots, t_{\ell(w_0)}^\circ)$ and applying the formula (9.9) lead to:

$$[b_1 \widehat{w_0}^{-1}]_0^{-\omega_k} = a^{\omega_{k^\star}} \prod_{\ell} t_\ell^{\langle \alpha_{|i_\ell|}^\vee, w_0 \geq \ell^{-1} \omega_k \rangle} \quad \text{and} \quad (\widehat{w_0} [b \widehat{w_0}]_0^{-1} \widehat{w_0}^{-1})^{\omega_k} = a'^{-\omega_{k^\star}} \prod_{\ell} t_\ell^{\langle \alpha_{|i_\ell|}^\vee, w_0 \geq \ell^{-1} \omega_k \rangle}.$$

So we have the following equality.

$$(9.10) \quad ([b_1]_0 [b_1 \widehat{w_0}^{-1}]_0^{-1} [b]_0 \widehat{w_0} [b \widehat{w_0}]_0^{-1} \widehat{w_0}^{-1})^{\omega_k} = \prod_{\ell} (t_\ell^\star t_\ell^\circ)^{-1} \langle \alpha_{|i_\ell|}^\vee, w_0 \geq \ell^{-1} \omega_k \rangle.$$

Now, let us fix some $\mathbf{j} = j_1 \dots j_{\ell(w_0)} \in R(1, w_0)$ and evaluate some $b_1, b \in N \cap G^{e, w_0}$ by $\mathbf{z}, \mathbf{z}' \in \mathcal{X}_1$, that is: $b_1 = \text{ev}_{\mathbf{j}}(\mathbf{z})$ and $b = \text{ev}_{\mathbf{j}}(\mathbf{z}')$. Let us also denote $\mathbf{j}_{\ell} := j_1 \dots j_{\ell}$ for every $\ell \leq \ell(w_0)$. We obtain from the relations $\mathbf{x}_{\mathbf{j}}(t_1, \dots, t_{\ell(w_0)}) = \text{ev}_{\mathbf{j}}(\mathbf{z})$ and $\mathbf{x}_{\mathbf{j}}(t'_1, \dots, t'_{\ell(w_0)}) = \text{ev}_{\mathbf{j}}(\mathbf{z}')$ the equalities:

$$t_{\ell} = \prod_{j < N^{j_{\ell}}(\mathbf{j}_{\ell})} z_{\binom{j_{\ell}}{j}} \quad \text{and} \quad t'_{\ell} = \prod_{j < N^{j_{\ell}}(\mathbf{j}_{\ell})} z'_{\binom{j_{\ell}}{j}}.$$

Moreover, because $b \in N$, we have $\pi_1(\mathbf{z}') = 1$. Thus, adding Lemma 5.15, the formula (5.5) and Proposition 5.17 to the previous equalities gives:

$$t_{\ell}^{\star} t_{\ell}^{\odot -1} = - \prod_{j < N^{j_{\ell}}(\mathbf{j}_{\ell})} (z_{\binom{j_{\ell}}{j}} z'_{\binom{j_{\ell}}{j}})^{-1} = - \prod_{j < N^{j_{\ell}}(\mathbf{j}_{\ell})} z_{\binom{j_{\ell}}{j}}^{-1} z'_{\binom{j_{\ell}}{j}}.$$

In fact, it is easy to see that the same kind of formula can be obtained when the evaluation of b_1 and b are done respectively for any \mathbf{i}_1 and \mathbf{i}_2 in $R(1, w_0)$. The setting $\mathbf{i} = \mathbf{i}_1 \mathbf{i}_2 \in D_e(w_0)$ and the equality $b_1 b H = \text{ev}_{\mathbf{i}}^{\text{red}}(\mathbf{x})$ then imply that $\mathbf{x}^{\text{red}} = \mathbf{m}(\mathbf{z}, \mathbf{z}'^{\text{red}})$. We finally apply Proposition 9.6 and the equality (2.3) on the formula (9.10) to end the proof of Proposition 9.7, that is:

$$\begin{aligned} X_i &= x_{\binom{i}{N^i(\mathbf{i})}} \prod_{k=1}^l \prod_{\ell=1}^{\ell(w_0)} \prod_{j < N^{i_{\ell}}(\mathbf{i}_{1\ell}), j' < N^{i_{\ell}}(\mathbf{i}_{2\ell})} (-z_{\binom{i_{\ell}}{j}}^{-1} z'_{\binom{i_{\ell}}{j'}})^{(A^{-1})_{ki} \langle \alpha_{i_{\ell}}^{\vee}, w_0 \sum_{\ell}^{-1} \omega_k \rangle} \\ &= x_{\binom{i}{N^i(\mathbf{i})}} \prod_{k=1}^l \prod_{\ell=1}^{\ell(w_0)} \prod_{j_1 < N^{i_{\ell}}(\mathbf{i}_{1\ell}), j_2 < N^{i_{\ell}}(\mathbf{i}_{2\ell})} (-x_{\binom{i_{\ell}}{j_1}}^{-1} x_{\binom{i_{\ell}}{N^{i_{\ell}}(\mathbf{i}_{1\ell}) + j_2}})^{(A^{-1})_{ki} \langle \alpha_{i_{\ell}}^{\vee}, w_0 \sum_{\ell}^{-1} \omega_k \rangle}. \end{aligned}$$

10. AN ELEMENTARY APPROACH FOR THE CASE $G = \text{SL}(2, \mathbb{C})$.

To fix the ideas, we consider with full details all the evaluation maps met before, and the related cluster combinatorics, in the simplest case: the case $G = \text{SL}(2, \mathbb{C})$. We thus start by recalling the construction of Fock and Goncharov for $(\text{SL}(2, \mathbb{C}), \pi_G)$ and successively consider the models $(\text{SL}(2, \mathbb{C}), \pi_*)$ and $(\text{SL}(2, \mathbb{C})^*, \pi_{G^*})$ for dual Poisson-Lie groups. And, as a conclusion, we give the quantization of this elementary construction by considering the cluster combinatorics associated with the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. This section is written to be as self-contained as possible.

10.1. Elementary Lie data. Let us recall that the complex simple Lie group

$$(10.1) \quad \text{SL}(2, \mathbb{C}) = \left\{ \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} : t_{11}t_{22} - t_{12}t_{21} = 1, \quad t_{ij} \in \mathbb{C} \right\}.$$

has its Lie algebra \mathfrak{g} equal to the set $\mathfrak{sl}(2, \mathbb{C})$ of 2-squared complex matrices which have a zero trace. The Chevalley generators $\{e_1, f_1, h_1\}$ and its related basis $\{e_1, f_1, h^1\}$ are then given by the following matrices:

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad h^1 = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}.$$

Using the exponential map $\exp : \mathfrak{g} \rightarrow G$, which, in this case, associates to a matrix $M \in \mathfrak{g}$ the usual matrix $\sum_{n=0}^{\infty} \frac{M^n}{n!} \in G$, we get the following generators of G , the two last ones

being associated to every non-zero complex number x .

$$E^1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad F^1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad H_1(x) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \quad H^1(x) = \begin{pmatrix} x^{1/2} & 0 \\ 0 & x^{-1/2} \end{pmatrix}.$$

In particular, these generators of the diagonal subgroup H of G satisfy the relation $H^1(x^2) = H_1(x)$ for every complex number $x \in \mathbb{C}_{\neq 0}$, which agrees with the formula (2.3), because the Cartan matrix A is simply here the number 2. Let us stress, however, that the generator $H^1(x)$ is generally ill-defined on $\mathrm{SL}(2, \mathbb{C})$. It is because $\mathrm{SL}(2, \mathbb{C})$ is not of adjoint type, but simply connected. The related adjoint group is $\mathrm{PGL}(2, \mathbb{C})$, and $H^1(x)$ is well-defined on $\mathrm{PGL}(2, \mathbb{C})$, because of the following identity.

$$H^1(x) = \begin{pmatrix} x^{1/2} & 0 \\ 0 & x^{-1/2} \end{pmatrix} \stackrel{\mathrm{PGL}(2, \mathbb{C})}{=} \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}.$$

Now, because there is only one simple root α_1 , the Weyl group W contains only two elements $\{1, s_1\}$ and the different double reduced words are the double words $1, \bar{1}, 1\bar{1}, \bar{1}1$ without forgetting the trivial double word $\mathbf{1}$ associated to the unity element of the direct product $W \times W$. Finally, the r -matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$ associated to $\mathfrak{sl}(2, \mathbb{C})$ and its related elements $r_{\pm} \in \mathfrak{g} \otimes \mathfrak{g}$ are given by the following formulas.

$$(10.2) \quad r = e_1 \wedge f_1, \quad r_+ = \frac{1}{4}h_1 \otimes h_1 + e_1 \otimes f_1 \quad \text{and} \quad r_- = -\frac{1}{4}h_1 \otimes h_1 - f_1 \otimes e_1.$$

10.2. Cluster \mathcal{X} -varieties related to $(\mathrm{SL}(2, \mathbb{C}), \pi_G)$. The evaluation maps of Fock and Goncharov associated to the previous double reduced words are then the following:

$$\begin{aligned} \mathrm{ev}_{\mathbf{1}}(x_0) &= H^1(x_0) \in G^{1,1} \\ &= \begin{pmatrix} x_0^{1/2} & 0 \\ 0 & x_0^{-1/2} \end{pmatrix}. \\ \\ \mathrm{ev}_1(y_0, y_1) &= H^1(y_0)E^1H^1(y_1) \in G^{1, w_0} \\ &= \begin{pmatrix} y_0^{1/2}y_1^{1/2} & y_0^{1/2}y_1^{-1/2} \\ 0 & y_0^{-1/2}y_1^{-1/2} \end{pmatrix} = \begin{pmatrix} 1 & y_0 \\ 0 & 1 \end{pmatrix} H^1(y_0y_1). \\ \\ \mathrm{ev}_{\bar{1}}(z_0, z_1) &= H^1(z_0)F^1H^1(z_1) \in G^{w_0, 1} \\ &= \begin{pmatrix} z_0^{1/2}z_1^{1/2} & 0 \\ z_0^{-1/2}z_1^{1/2} & z_0^{-1/2}z_1^{-1/2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z_0^{-1} & 1 \end{pmatrix} H^1(z_0z_1). \\ \\ \mathrm{ev}_{1\bar{1}}(u_0, u_1, u_2) &= H^1(u_0)E^1H^1(u_1)F^1H^1(u_2) \in G^{w_0, w_0} \\ &= \begin{pmatrix} u_0^{1/2}u_1^{1/2}u_2^{1/2} + u_0^{1/2}u_1^{-1/2}u_2^{1/2} & u_0^{1/2}u_1^{-1/2}u_2^{-1/2} \\ u_0^{-1/2}u_1^{1/2}u_2^{1/2} & u_0^{-1/2}u_1^{-1/2}u_2^{-1/2} \end{pmatrix} \\ &= \begin{pmatrix} 1 + u_1^{-1} & u_0 \\ u_0^{-1}u_1^{-1} & 1 \end{pmatrix} H^1(u_0u_1u_2). \\ \\ \mathrm{ev}_{\bar{1}1}(v_0, v_1, v_2) &= H^1(v_0)F^1H^1(v_1)E^1H^1(v_2) \in G^{w_0, w_0} \\ &= \begin{pmatrix} v_0^{1/2}v_1^{1/2}v_2^{1/2} & v_0^{1/2}v_1^{1/2}v_2^{-1/2} \\ v_0^{-1/2}v_1^{1/2}v_2^{1/2} & v_0^{-1/2}v_1^{-1/2}v_2^{-1/2} + v_0^{-1/2}v_1^{1/2}v_2^{-1/2} \end{pmatrix} \\ &= \begin{pmatrix} 1 & v_0v_1 \\ v_0^{-1} & v_1 + 1 \end{pmatrix} H^1(v_0v_1v_2). \end{aligned}$$

Again, the reader annoyed with the rational powers is free to replace $\mathrm{SL}(2, \mathbb{C})$ by $\mathrm{PGL}(2, \mathbb{C})$. Let us remark, however, that for every $u, v \in W$ and every double reduced word $\mathbf{i} \in R(u, v)$, the associated reduced evaluation maps $\mathrm{ev}_{\mathbf{i}}^{\mathrm{red}} : \mathcal{X}_{\mathbf{i}}^{\mathrm{red}} \rightarrow G^{u,v}/H$ described in Subsection 4.1 are well-defined birational isomorphisms both on $\mathrm{SL}(2, \mathbb{C})$ and $\mathrm{PGL}(2, \mathbb{C})$. Moreover, let us notice that it is also possible to construct the two last evaluation maps from the others, using the amalgamated product. Indeed, according to the formula (3.4), we get the relations

$$\begin{aligned} u_0 &:= y_0, & u_1 &:= y_1 z_0, & u_2 &:= z_1 \\ \text{and} \\ v_0 &:= z_0, & v_1 &:= z_1 y_0, & v_2 &:= y_1. \end{aligned}$$

From the other hand, if $\mathrm{ev}_{1\bar{1}}(u_0, u_1, u_2) = \mathrm{ev}_{\bar{1}1}(v_0, v_1, v_2)$, then we have the following relations between the u_i and the v_j :

$$(10.3) \quad \begin{cases} v_0 = u_0(1 + u_1) \\ v_1 = u_1^{-1} \\ v_2 = u_2(1 + u_1) \end{cases} \quad \text{and} \quad \begin{cases} u_0 = v_0(1 + v_1^{-1})^{-1} \\ u_1 = v_1^{-1} \\ u_2 = v_2(1 + v_1^{-1})^{-1} \end{cases}.$$

Now, for every $i, j \in [1, 2]$, let t_{ij} be the coordinate function associated to (10.1). Applying the formula (10.2) to the Sklyanin bracket (2.8), we can see that the standard Poisson bracket on the Poisson-Lie group G is given by the following equalities:

$$\begin{cases} \{t_{11}, t_{12}\}_G = \frac{1}{2}t_{11}t_{12}, & \{t_{11}, t_{21}\}_G = \frac{1}{2}t_{11}t_{21}, \\ \{t_{11}, t_{22}\}_G = t_{12}t_{21}, & \{t_{12}, t_{21}\}_G = 0, \\ \{t_{12}, t_{22}\}_G = \frac{1}{2}t_{12}t_{22}, & \{t_{21}, t_{22}\}_G = \frac{1}{2}t_{21}t_{22}. \end{cases}$$

We quickly check that the maps ev_1 , ev_1 , and $\mathrm{ev}_{\bar{1}}$ are Poisson when the matrices (resp. quivers) establishing the Poisson structure on the seed \mathcal{X} -tori is given respectively by:

$$\varepsilon(\mathbf{1}) = (0), \quad \varepsilon(1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varepsilon(\bar{1}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$



Then the amalgamation procedure leads to the following the matrices (resp. quivers) establishing the Poisson structures on the associated seed \mathcal{X} -tori for which the maps $\mathrm{ev}_{1\bar{1}}$ and $\mathrm{ev}_{\bar{1}1}$ are Poisson:

$$(10.4) \quad \varepsilon(1\bar{1}) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \varepsilon(\bar{1}1) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$



Looking at them, it is clear that the expressions in (10.3) describe respectively the cluster transformation $\mu_{1\bar{1} \rightarrow \bar{1}1} : \mathcal{X}_{1\bar{1}} \rightarrow \mathcal{X}_{\bar{1}1}$ associated to the variable u_1 and the cluster transformation $\mu_{\bar{1}1 \rightarrow 1\bar{1}} : \mathcal{X}_{\bar{1}1} \rightarrow \mathcal{X}_{1\bar{1}}$ associated to v_1 :

$$(v_0, v_1, v_2) = \mu_{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}(u_0, u_1, u_2) \quad \text{and} \quad (u_0, u_1, u_2) = \mu_{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}(v_0, v_1, v_2).$$

$$\begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{\text{ev}_1} & (G^{1,1}, \pi_G) \\ & & \\ \mathcal{X}_{\overline{1}} & \xrightarrow{\text{ev}_{\overline{1}}} & (G^{w_0,1}, \pi_G) \\ & & \\ \mathcal{X}_{1\overline{1}} & \xrightarrow{\text{ev}_{1\overline{1}}} & (G^{w_0,w_0}, \pi_G) \\ & & \\ \mathcal{X}_{\overline{1}1} & \xrightarrow{\text{ev}_{\overline{1}1}} & (G^{w_0,w_0}, \pi_G) \end{array}$$
$$\begin{cases} \{t_{11}, t_{12}\}_* = t_{12}t_{22}, & \{t_{11}, t_{21}\}_* = -t_{21}t_{22}, \\ \{t_{11}, t_{22}\}_* = 0, & \{t_{12}, t_{21}\}_* = t_{11}t_{22} - t_{22}^2, \\ \{t_{12}, t_{22}\}_* = t_{12}t_{22}, & \{t_{21}, t_{22}\}_* = -t_{21}t_{22}. \end{cases}$$
$$\begin{aligned} \text{ev}_1^{\text{dual}}(x_0, t) &= H^1(x_0)E^1\widehat{w_0}H_1(t)(F^1)^{-1}H^1(x_0^{-1}) \in F_{w_0} \\ &= \begin{pmatrix} t+t^{-1} & -x_0t^{-1} \\ x_0^{-1}t & 0 \end{pmatrix}. \end{aligned}$$
$$\begin{aligned} \text{ev}_{1\overline{1}}^{\text{dual}}(y_0, y_1, t) &= H^1(y_0)F^1 \text{ev}_1^{\text{dual}}(y_1, t) (F^1)^{-1}H^1(y_0^{-1}) \\ &= \begin{pmatrix} t^{-1}(1+y_1)+t & -y_0y_1t^{-1} \\ y_0^{-1}(t(1+y_1^{-1})+t^{-1}(1+y_1)) & -y_1t^{-1} \end{pmatrix}, \\ \text{ev}_{1\overline{1}}^{\text{dual}}(\tilde{y}_0, \tilde{y}_1, t) &= \begin{pmatrix} t^{-1/2}(1+\tilde{y}_1^{-1})+t & -t^{-1}\tilde{y}_0(1+\tilde{y}_1^{-1}) \\ \tilde{y}_0^{-1}(t+t^{-1}\tilde{y}_1^{-1}) & -\tilde{y}_1^{-1}t \end{pmatrix}. \end{aligned}$$

These evaluations are particular cases of the twisted evaluations described in Subsection 5.4. The remaining twisted evaluations $\widehat{\text{ev}}_{11}, \widehat{\text{ev}}_{1\overline{1}} : \mathcal{X}_{[11]_{\mathfrak{R}}} \rightarrow BB_-$ and $\widehat{\text{ev}}_{\overline{1}\overline{1}} : \mathcal{X}_{[\overline{1}\overline{1}]_{\mathfrak{R}}} \rightarrow BB_-$, which are described in Subsection 7.2, also parameterize the variety BB_- . They are given by the following formulas:

$$\begin{aligned}
\widehat{\text{ev}}_{11}(z_0, z_1, t) &= \widehat{\text{ev}}_{1\bar{1}}(z_0, z_1, t) \\
&= H^1(z_0)E^1 \widehat{\text{ev}}_1(z_1, t) (E^1)^{-1}H^1(z_0^{-1}) \\
&= \begin{pmatrix} (1+z_1^{-1})t+t^{-1} & -z_0((1+z_1^{-1})t+(1+z_1)t^{-1}) \\ z_0^{-1}z_1^{-1}t & -z_1^{-1}t \end{pmatrix} \\
\widehat{\text{ev}}_{\bar{1}\bar{1}}(y_0, y_1, t) &= \text{ev}_{\bar{1}\bar{1}}^{\text{dual}}(y_0, y_1, t) \\
&= \begin{pmatrix} t^{-1}(1+y_1)+t & -t^{-1}y_0y_1 \\ y_0^{-1}(t(1+y_1^{-1})+t^{-1}(1+y_1)) & -y_1t^{-1} \end{pmatrix}
\end{aligned}$$

It is easy to check that all these maps are Poisson when the matrices (resp. quivers) establishing the Poisson structure on the related seed \mathcal{X} -tori are given respectively by the matrices (resp. quivers):

$$(10.5) \quad \eta(11) = \eta(1\bar{1}) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \eta(\bar{1}1) = \eta(\bar{1}\bar{1}) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\circ \xrightarrow{\bullet} \bullet \quad \circ \quad \quad \quad \circ \xrightarrow{\bullet} \bullet \quad \circ$$

Therefore, the truncation map (4.5) gives the way to pass from the Poisson structures defined by (10.4) to the Poisson structures defined by (10.5). We thus get a cluster \mathcal{X} -variety, denoted $\mathcal{X}_{e \leq e}$, for the variety F_{w_0} and two isomorphic cluster \mathcal{X} -varieties for the variety BB_- , denoted \mathcal{X}_e and \mathcal{X}_{w_0} , and respectively associated to the cluster variables (y_0, y_1, t) and (z_0, z_1, t) .

10.3.2. *Remarks about evaluations maps for $(\text{PGL}(2, \mathbb{C}), \pi_*)$.* The careful reader will have noticed that the evaluation maps related to $(\text{SL}(2, \mathbb{C}), \pi_*)$ we have just obtained slightly differ from the twisted evaluation maps of Section 7. Again, it is because the Lie group $\text{SL}(2, \mathbb{C})$ is not of adjoint type. According to Remark 7.14, the corresponding evaluation maps for $G = \text{PGL}(2, \mathbb{C})$ are the following.

$$\begin{aligned}
\underline{\text{ev}}_1^{\text{dual}}(x_0, t) &= H^1(x_0)E^1\widehat{w_0}H^1(t)(F^1)^{-1}H^1(x_0^{-1}) \in F_{w_0} \\
&= \begin{pmatrix} t^{1/2}+t^{-1/2} & -x_0t^{-1/2} \\ x_0^{-1}t^{1/2} & 0 \end{pmatrix}.
\end{aligned}$$

$$\begin{aligned}
\underline{\text{ev}}_{\bar{1}\bar{1}}^{\text{dual}}(y_0, y_1, t) &= H^1(y_0)F^1 \text{ev}_1^{\text{dual}}(y_1, t) (F^1)^{-1}H^1(y_0^{-1}) \\
&= \begin{pmatrix} t^{-1/2}(1+y_1)+t^{1/2} & -y_0y_1t^{-1/2} \\ y_0^{-1}(t^{1/2}(1+y_1^{-1})+t^{-1/2}(1+y_1)) & -y_1t^{-1/2} \end{pmatrix},
\end{aligned}$$

$$\underline{\text{ev}}_{1\bar{1}}^{\text{dual}}(\widetilde{y}_0, \widetilde{y}_1, t) = \begin{pmatrix} t^{-1/2}(1+\widetilde{y}_1^{-1/2})+t^{1/2} & -t^{-1/2}\widetilde{y}_0(1+\widetilde{y}_1^{-1}) \\ \widetilde{y}_0^{-1}(t^{1/2}+t^{-1/2}\widetilde{y}_1^{-1}) & -\widetilde{y}_1^{-1}t^{1/2} \end{pmatrix}.$$

$$\begin{aligned}
\widehat{\text{ev}}_{11}(z_0, z_1, t) &= \widehat{\text{ev}}_{1\overline{1}}(z_0, z_1, t) \\
&= H^1(z_0)E^1 \widehat{\text{ev}}_1(z_1, t) (E^1)^{-1}H^1(z_0^{-1}) \\
&= \begin{pmatrix} (1+z_1^{-1})t^{1/2} + t^{-1/2} & -z_0((1+z_1^{-1})t^{1/2} + (1+z_1)t^{-1/2}) \\ z_0^{-1}z_1^{-1}t^{1/2} & -z_1^{-1}t^{1/2} \end{pmatrix} \\
\widehat{\text{ev}}_{\overline{1}\overline{1}}(y_0, y_1, t) &= \text{ev}_{\overline{1}\overline{1}}^{\text{dual}}(y_0, y_1, t) \\
&= \begin{pmatrix} t^{-1/2}(1+y_1) + t^{1/2} & -t^{-1/2}y_0y_1 \\ y_0^{-1}(t^{1/2}(1+y_1^{-1}) + t^{-1/2}(1+y_1)) & -y_1t^{-1/2} \end{pmatrix}
\end{aligned}$$

10.3.3. *How to use the saltation map.* If the evaluations $\text{ev}_{\overline{1}\overline{1}}^{\text{dual}}(y_0, y_1, t)$ and $\widehat{\text{ev}}_{11}(z_0, z_1, t)$ parameterize the same element, we quickly check with the expressions above that the map $\varphi : (y_0, y_1, t) \mapsto (z_0, z_1, t)$ is given by:

$$(10.6) \quad \begin{cases} z_0 &= y_0(1+y_1^{-1})^{-1}(1+y_1^{-1}t^2)^{-1} \\ z_1 &= t^2y_1^{-1} \end{cases}.$$

Before to link the map φ with the cluster combinatorics we have developed, let us stress (again) that saltations are really needed in the story because, the variable t being a Casimir function, you cannot expect to obtain a formula such as $z_1 = t^2y_1^{-1}$ by only cluster transformation. Now, let us remark that the cover $\mathfrak{p}_{\mathcal{X}} : \mathcal{X}_{[\overline{1}\overline{1}]_{\mathfrak{R}}} \rightarrow \mathcal{X}_{[\overline{1}\overline{1}]_{\mathfrak{R}}}$ is of degree 2 and given by the formula

$$\mathfrak{p}_{\mathcal{X}} : (y_1, y_2, t) \mapsto (y_1, y_2, t^2).$$

We are going to prove that the equality $\mathfrak{p}_{\mathcal{X}} \circ \varphi = \Xi_{s_1} \circ \mathfrak{p}_{\mathcal{X}}$ is satisfied, where Ξ_{s_1} denotes the birational Poisson isomorphism given by Corollary 8.9. To get it, let us first describe the saltation Ξ_1 given by (8.8). It is associated to the following generalized cluster transformation, acting on every element $(x_0, x_1, x_2) \in \mathcal{X}_{1\overline{1}}$.

$$\begin{aligned}
\overline{\mu}_{\overline{1}\overline{1}} \circ \zeta_1 \circ \mu_{1\overline{1} \rightarrow \overline{1}\overline{1}}(x_0, x_1, x_2) &= \mu_{\binom{1}{1}} \circ \mu_{\binom{1}{2}} \circ \mu_{\binom{1}{1}}(x_0, x_1, x_2) \\
&= \mu_{\binom{1}{1}} \circ \mu_{\binom{1}{2}}(x_0(1+x_1), x_1^{-1}, x_2(1+x_1)) \\
&= \mu_{\binom{1}{1}}(x_0(1+x_1), x_1^{-1}, x_2^{-1}(1+x_1)^{-1}) \\
&= (x_0, x_1, x_2^{-1}x_1^{-1}).
\end{aligned}$$

$$\text{Therefore} \quad \Xi_1(x_0, x_1, t) = (x_0, x_1^{-1}t^{-1}, t),$$

$$\text{because} \quad \Xi_1 \circ \mathfrak{t}_{\binom{1}{2}}(t) = \mathfrak{t}_{\binom{1}{1}}(t) \circ \mu_{\binom{1}{1}} \circ \mu_{\binom{1}{2}} \circ \mu_{\binom{1}{1}}.$$

We then get the following formula for the birational isomorphism Ξ_{s_1} .

$$\begin{aligned}
 \Xi_{s_1}(y_0, y_1, t) &= \mu_{[\bar{1}\bar{1}]_{\mathfrak{R}} \rightarrow [\bar{1}\bar{1}]_{\mathfrak{R}}} \circ \Xi_1 \circ \mu_{[\bar{1}\bar{1}]_{\mathfrak{R}} \rightarrow [1\bar{1}]_{\mathfrak{R}}}(y_0, y_1, t) \\
 &= \mu_{[\bar{1}\bar{1}]_{\mathfrak{R}} \rightarrow [\bar{1}\bar{1}]_{\mathfrak{R}}} \circ \Xi_1(y_0(1 + y_1^{-1})^{-1}, y_1^{-1}, t) \\
 (10.7) \quad &= \mu_{[\bar{1}\bar{1}]_{\mathfrak{R}} \rightarrow [\bar{1}\bar{1}]_{\mathfrak{R}}}(y_0(1 + y_1^{-1})^{-1}, y_1 t^{-1}, t) \\
 &= (y_0(1 + y_1^{-1})^{-1}(1 + y_1^{-1}t)^{-1}, y_1^{-1}t, t).
 \end{aligned}$$

And it is now clear that the equality $\mathfrak{p}_{\mathcal{X}} \circ \varphi = \Xi_{s_1} \circ \mathfrak{p}_{\mathcal{X}}$ is satisfied. Moreover, the way the cluster varieties \mathcal{X}_e and \mathcal{X}_{s_1} are related by the saltation Ξ_1 is described by the elementary W -permutohedron pictured in Figure 31. Let us notice, for the reader who prefers to

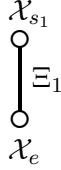


FIGURE 31. Saltation and cluster \mathcal{X} -varieties for $\mathfrak{g} = A_1$

deal with $\mathrm{PGL}(2, \mathbb{C})$, that things are much simpler for him because the covering $\mathfrak{p}_{\mathcal{X}}$ is then the identity map on seed \mathcal{X} -tori while the formulas for the saltation and cluster transformations remain unchanged.

10.4. Evaluation maps for $(\mathrm{SL}(2, \mathbb{C})^*, \pi_{G^*})$. We first recall that the set $\mathrm{SL}(2, \mathbb{C})^*$ has the following description

$$\begin{aligned}
 (10.8) \quad \mathrm{SL}(2, \mathbb{C})^* &= \left\{ \left(\begin{pmatrix} t_{11}^+ & t_{12}^+ \\ 0 & t_{22}^+ \end{pmatrix}, \begin{pmatrix} t_{11}^- & 0 \\ t_{21}^- & t_{22}^- \end{pmatrix} \right) \right. \\
 &\quad \left. : t_{11}^- = (t_{11}^+)^{-1}, \quad t_{22}^- = (t_{22}^+)^{-1}, \quad t_{11}^+ t_{22}^+ = 1, \quad t_{ij}^\pm \in \mathbb{C} \right\}.
 \end{aligned}$$

Now, let us recall its Poisson structure π_{G^*} . Again, we denote t_{ij}^\pm the corresponding coordinate functions. The Poisson bracket we are looking for on $\mathrm{SL}(2, \mathbb{C})^*$ is given by the following equalities:

$$\begin{cases} \{t_{11}^\pm, t_{12}^\pm\}_{G^*} = \pm t_{11}^\pm t_{12}^\pm, & \{t_{11}^\pm, t_{21}^\pm\}_{G^*} = \mp t_{11}^\pm t_{21}^\pm, \\ \{t_{11}^\pm, t_{22}^\pm\}_{G^*} = 0, & \{t_{12}^\pm, t_{21}^\pm\}_{G^*} = t_{22}^\pm t_{11}^\pm, \\ \{t_{12}^\pm, t_{22}^\pm\}_{G^*} = \pm t_{12}^\pm t_{22}^\pm, & \{t_{21}^\pm, t_{22}^\pm\}_{G^*} = \mp t_{21}^\pm t_{22}^\pm. \end{cases}$$

Because of the Poisson covering $\phi : (\mathrm{SL}(2, \mathbb{C})^*, \pi_{G^*}) \rightarrow (\mathrm{SL}(2, \mathbb{C}), \pi_*)$ of degree 2, we use the previous evaluation maps on $(\mathrm{SL}(2, \mathbb{C}), \pi_*)$ to get the evaluation maps on $(\mathrm{SL}(2, \mathbb{C})^*, \pi_{G^*})$. To do that, let us introduce the following notations for every non-zero complex number x .

$$E^1(x) = H^1(x)E^1H^1(x^{-1}) \quad \text{and} \quad F^1(x) = H^1(x^{-1})F^1H^1(x).$$

Now, if the evaluations $\mathrm{ev}_{11}^{\mathrm{dual}}(y_0, y_1, t)$ and $\widehat{\mathrm{ev}}_{11}(z_0, z_1, t)$ gives the same element $nh^2n_-^{-1}$ such that n (resp. n_-) is an upper (resp. lower) triangular matrix with the number 1 on

the diagonal entries and h a diagonal matrix, then we can evaluate h , n and n_- is the following way, using for example a computation in the spirit of Lemma 9.5.

$$\begin{cases} h = H^1(-t^{-1}z_1) = H^1(-ty_1^{-1}) \\ n = [[F^1(y_0) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}]]_+ = E^1(y_0) \\ n_- = [E^1(z_0) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}]_- = F^1(z_0^{-1}) . \end{cases}$$

And because the map Ξ_{s_1} is such that the equality $\mathbf{p}_{\mathcal{X}} \circ \varphi = \Xi_{s_1} \circ \mathbf{p}_{\mathcal{X}}$ is satisfied, where $\varphi : (y_0, y_1, t) \mapsto (z_0, z_1, t)$, these formulas are in agreement with Theorem 9.12 which states that:

$$\begin{cases} \text{ev}_{\overline{11}}^+(\mathbf{y}) = E^1(y_0)H^1(-ty_1^{-1}) \\ \text{ev}_{\overline{11}}^-(\mathbf{y}) = F^1(\Xi_{s_1}(\mathbf{y})_0^{-1})H^1(-ty_1^{-1})^{-1} \end{cases} \quad \text{and} \quad \begin{cases} \text{ev}_{11}^+(\mathbf{z}) = E^1(\Xi_{s_1}^{-1}(\mathbf{z})_0)H^1(-t^{-1}z_1) \\ \text{ev}_{11}^-(\mathbf{z}) = F^1(z_0^{-1})H^1(-t^{-1}z_1)^{-1} \end{cases} .$$

We finally the following description of G^* using matrices, analogous to the one given by (10.8), which involves our cluster variables and their relation $\varphi : (y_0, y_1, t) \mapsto (z_0, z_1, t)$ given by the relation (10.6).

$$\left(\begin{pmatrix} (-ty_1^{-1})^{1/2} & (-ty_1^{-1})^{-1/2}y_0 \\ 0 & (-ty_1^{-1})^{-1/2} \end{pmatrix}, \begin{pmatrix} (-ty_1^{-1})^{-1/2} & 0 \\ (-ty_1^{-1})^{-1/2}z_0^{-1} & (-ty_1^{-1})^{1/2} \end{pmatrix} \right) .$$

We are thus in an optimal position for the quantization process. Indeed, the R -matrix associated to the quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ associated to the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ and the related quantum group $\mathcal{F}_q(\text{SL}(2, \mathbb{C})^*)$ are respectively given by the following formulas.

$$(10.9) \quad \mathcal{R} = q^{\frac{1}{2}H \otimes H} E \otimes F .$$

$$L^+ = \begin{pmatrix} (qK)^{1/2} & (qK)^{1/2}(q - q^{-1})F \\ 0 & (qK)^{-1/2} \end{pmatrix} ,$$

$$L^- = \begin{pmatrix} (qK)^{-1/2} & 0 \\ (qK)^{-1/2}(q - q^{-1})E & (qK)^{1/2} \end{pmatrix} .$$

10.5. Quantum evaluation maps for $\mathcal{U}_q(\mathfrak{sl}(2, \mathbb{C}))$. The dual Poisson-Lie group (G^*, π_{G^*}) is the semi-classical limit of the quantum group $\mathcal{F}_q(G^*)$ which is isomorphic (as Hopf algebra) to the very famous quantized universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$. As a conclusion to this work, we give the quantum picture for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. For technical reasons, we consider the quantized universal enveloping algebra $\mathcal{U}_{q^{-1}}(\mathfrak{g})$ instead of $\mathcal{U}_q(\mathfrak{g})$. It is the $\mathbb{C}(q)$ -algebra generated by E , F , and K with relations

$$KE = q^{-2}EK, \quad KF = q^2FK, \quad \text{and} \quad EF - FE = \frac{K - K^{-1}}{q^{-1} - q} .$$

We now define the quantum tori $\mathcal{X}_{[\overline{11}]_{\mathfrak{g}}}^q$ and $\mathcal{X}_{[11]_{\mathfrak{g}}}^q$ as the $\mathbb{C}(q)$ -algebra generated respectively by the elements Y_0, Y_1, T and Z_0, Z_1, T with the q -commutation relations:

$$Y_0Y_1 = q^2Y_1Y_0 \quad \text{and} \quad Z_0Z_1 = q^{-2}Z_1Z_0 ;$$

$$Y_0T = TY_0 \quad \text{and} \quad Z_0T = TZ_0 ;$$

$$TY_1 = Y_1T \quad \text{and} \quad TZ_1 = Z_1T .$$

In particular, it is clear that the seed \mathcal{X} -tori $\mathcal{X}_{[\overline{11}]_{\mathfrak{R}}}$ and $\mathcal{X}_{[11]_{\mathfrak{R}}}$, whose Poisson structures are given by (10.5), are respectively the semi-classical limits of these quantum tori. Luckily, the quantum evaluation maps for $\mathcal{U}_{q^{-1}}(\mathfrak{g})$ come without effort from the semi-classical evaluation maps we have just obtained for $(\mathrm{SL}(2, \mathbb{C})^*, \pi_{G^*})$. (We don't forget to switch q into q^{-1} in the formula (10.9) according to the switch $\mathcal{U}_q(\mathfrak{g}) \leftrightarrow \mathcal{U}_{q^{-1}}(\mathfrak{g})$.) In particular, it is straightforward to check that the following map is an algebra morphism.

$$\left\{ \begin{array}{l} \mathrm{ev}^{q^{-1}}(K) = -qTY_1^{-1} = -qT^{-1}Z_1 ; \\ \mathrm{ev}^{q^{-1}}(F) = -(q^{-1} - q)^{-1}T^{-1}Y_1Y_0 ; \\ \mathrm{ev}^{q^{-1}}(E) = (q^{-1} - q)^{-1}Z_0^{-1} . \end{array} \right. \quad \text{if} \quad \left\{ \begin{array}{l} Z_0 = Y_0(1 + qY_1^{-1})^{-1}(1 + qY_1^{-1}T^2)^{-1} \\ Z_1 = T^2Y_1^{-1} \end{array} \right.$$

Moreover, the link between the Y_i and the Z_j is given by quantizing the birational Poisson isomorphism φ . To see that, we use the quantization formulas of [FG07a]: in the case $|\varepsilon_{ik}| = 1$, we get the following *quantum mutations*:

$$(10.10) \quad X_{\mu_k^q(i)} = \begin{cases} X_k^{-1} & \text{if } i = k; \\ X_i X_k^{[\varepsilon_{ik}]_+} (1 + qX_k)^{-\varepsilon_{ik}} & \text{if } i \neq k . \end{cases}$$

Therefore, the quantization of the computation (10.7) gives us the following equalities, using a still mysterious map which we denote Ξ_1^q .

$$\begin{aligned} \Xi_{s_1}^q(Y_0, Y_1, T) &= \mu_{[1\overline{1}]_{\mathfrak{R}} \rightarrow [\overline{11}]_{\mathfrak{R}}}^q \circ \Xi_1^q \circ \mu_{[\overline{11}]_{\mathfrak{R}} \rightarrow [1\overline{1}]_{\mathfrak{R}}}^q(Y_0, Y_1, T) \\ &= \mu_{[1\overline{1}]_{\mathfrak{R}} \rightarrow [\overline{11}]_{\mathfrak{R}}}^q \circ \Xi_1^q(Y_0(1 + qY_1^{-1})^{-1}, Y_1^{-1}, T) \\ &= \mu_{[1\overline{1}]_{\mathfrak{R}} \rightarrow [\overline{11}]_{\mathfrak{R}}}^q(Y_0(1 + qY_1^{-1})^{-1}, Y_1T^{-1}, T) \\ &= (Y_0(1 + qY_1^{-1})^{-1}(1 + qY_1^{-1}t)^{-1}, Y_1^{-1}T, T) . \end{aligned}$$

Moreover we have the equality $\mathfrak{p}_{\mathcal{X}} \circ \varphi^q = \Xi_{s_1}^q \circ \mathfrak{p}_{\mathcal{X}}$, where φ^q is the quantization of the map φ , that is $\varphi^q : (Y_0, Y_1, T) \rightarrow (Z_0, Z_1, T)$. Let us stress, however, that there is still something strange in this story. Indeed, by keeping the tropicalization formula, we therefore also get tropical quantum mutations from (10.10); we use them to introduce quantum saltation $\widetilde{\Xi}_1^q$, defined by intertwining generalized quantum cluster transformations with truncation maps, by mimicking the previous computation. We thus get:

$$\begin{aligned} \mu_{(1)}^q \circ \mu_{(2)}^q \circ \mu_{(1)}^q(X_0, X_1, X_2) &= \mu_{(1)}^q \circ \mu_{(2)}^q(X_0(1 + qX_1), X_1^{-1}, X_2(1 + qX_1)) \\ &= \mu_{(1)}^q(X_0(1 + qX_1), X_1^{-1}, X_2^{-1}(1 + qX_1)^{-1}) \\ &= (X_0, X_1, qX_2^{-1}X_1^{-1}) . \end{aligned}$$

$$\text{Therefore} \quad \widetilde{\Xi}_1^q(X_0, X_1, T) = (X_0, qX_1^{-1}T^{-1}, T) ,$$

$$\text{because} \quad \widetilde{\Xi}_1^q \circ \mathfrak{t}_{(2)(t)}^q = \mathfrak{t}_{(1)(t)}^q \circ \mu_{(1)}^q \circ \mu_{(2)}^q \circ \mu_{(1)}^q .$$

But, unfortunately, it is clear that we have $\widetilde{\Xi}_1^q \neq \Xi_1^q$.

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